

## DONALDSON INVARIANTS OF 4-MANIFOLDS WITH SIMPLE TYPE

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### 1. Introduction

The Donaldson invariant of a smooth simply connected 4-manifold  $X$  with odd  $b^+ \geq 3$  is a linear map

$$D_X : \mathbf{A}(X) = \text{Sym}_*(H_0(X) \oplus H_2(X)) \rightarrow \mathbf{R}$$

defined on the graded algebra  $\mathbf{A}(X)$ , where elements of  $H_i(X)$  are defined to have degree  $\frac{1}{2}(4-i)$ . Since its presentation by Simon Donaldson [8], this invariant has proven indispensable for distinguishing smooth 4-manifolds with the same homotopy type. Roughly, if  $x \in H_0(X)$ ,  $\alpha \in H_2(X)$ , and  $z = \alpha^a x^b \in \mathbf{A}(X)$  has degree  $d$ , one can define  $D_X$  by the formula

$$D_X(z) = \langle \mu(\alpha)^a \mu(x)^b, [\mathcal{M}_X^{2d}] \rangle,$$

where  $[\mathcal{M}_X^{2d}]$  is the fundamental class of the (compactified)  $2d$ -dimensional moduli space of anti-self-dual connections on an  $SU(2)$  bundle over  $X$ , and  $\mu : H_*(X) \rightarrow H^{4-*}(\mathcal{M}_X^{2d})$  is a canonical homomorphism. The instanton moduli spaces  $\mathcal{M}_X^{2d}$  have formal dimensions congruent to  $-3(1 + b_X^+) \pmod{4}$ , and  $D_X$  is defined to be 0 in degrees other than  $\frac{1}{2}(1 + b_X^+) \pmod{4}$ .

Despite its utility, the Donaldson invariant has proven difficult to evaluate, and its general form has remained elusive. In this paper we investigate the general structure of this invariant through a study of its behavior in the presence of embedded spheres. A turning point in the study of the invariants arose with the results of P. Kronheimer and T. Mrowka [24] concerning the structure of the Donaldson invariants under the technical assumption of "simple type." This assumption states essentially that for the generator  $x$  of  $H_0(X)$  and arbitrary  $z \in \mathbf{A}(X)$ ,  $D_X(x^2 z) = 4 D_X(z)$ . Their results are obtained through a study of

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connections with singularities at an embedded surface of positive self-intersection. It turns out that our study also gives a rather full understanding of the form of the Donaldson invariants, as well as determining strong lower bounds for the number of positive double points for immersed representatives of 2-dimensional homology classes, for simply connected smooth 4-manifolds with simple type. Further, we show that many manifolds do indeed have simple type; in particular all elliptic surfaces, complete intersections, and certain branched covers of  $\mathbf{CP}^2$  and  $S^2 \times S^2$  have simple type (cf. Theorem 3.9). To prove our results we use gauge-theoretic “neck-stretching” arguments which involve splitting a 4-manifold along a 3-manifold and analyzing how the moduli spaces of anti-self-dual connections over the 4-manifold decompose into pieces. The complexity of these arguments is proportional to the complexity of the character variety of the 3-manifold along which the 4-manifold is split. In this paper we split along the most elementary 3-manifolds, the lens spaces  $L(p, 1)$ , with the most elementary nontrivial character variety, a finite collection of points. This accounts for the relative simplicity of the arguments of §4.

Given a simply connected 4-manifold  $X$  of simple type we study the formal power series  $\mathbf{D}_X : H_2(X) \rightarrow \mathbf{R}$ , the Donaldson series of  $X$  [24], defined by

$$\mathbf{D}_X(\alpha) = D_X\left(\left(1 + \frac{x}{2}\right) \exp(\alpha)\right).$$

It is also convenient to define the formal power series  $\mathbf{K}_X = \exp(-Q/2)\mathbf{D}_X$  on  $H_2(X)$ , where  $Q$  is the intersection form of  $X$ . There are also Donaldson invariants corresponding to  $SO(3)$  instanton moduli spaces and corresponding formal series. These depend on an  $SO(3)$  bundle  $P$  over  $X$  and an integral lift  $c \in H_2(X; \mathbf{Z})$  of (the Poincaré dual of) the second Stiefel-Whitney class  $w_2(P)$ . The corresponding invariants are denoted  $D_{X,c}$  and  $\mathbf{D}_{X,c}$ . The structure theorem of Kronheimer and Mrowka is:

**Theorem** (Kronheimer and Mrowka [24], [25]). *Let  $X$  be a simply connected 4-manifold of simple type. Then the following hold:*

(i) *There exist finitely many homology classes  $\kappa_1, \dots, \kappa_p \in H_2(X, \mathbf{Z})$  and nonzero rational numbers  $a_1, \dots, a_p$  such that*

$$\mathbf{D}_X = \exp(Q/2) \sum_{s=1}^p a_s e^{\kappa_s}$$

*as analytic functions on  $H_2(X)$ . Each of the ‘basic classes’  $\kappa_s$  is characteristic, i.e.,  $\kappa_s \cdot x \equiv x \cdot x \pmod{2}$  for all  $x \in H_2(X; \mathbf{Z})$ .*

Further, suppose  $c \in H_2(X; \mathbf{Z})$ . Then

$$D_{X,c} = \exp(Q/2) \sum_{s=1}^p (-1)^{(c^2 + \kappa_s \cdot c)/2} a_s e^{\kappa_s}.$$

(ii) If  $u \in H_2(X; \mathbf{Z})$  is represented by an embedded surface of genus  $g$  with self-intersection  $u^2 \geq 0$ , then for each  $s$

$$2g - 2 \geq u^2 + |\kappa_s \cdot u|.$$

Here we view the homology class  $\kappa_s$  as acting on an arbitrary homology class by intersection, i.e.,  $\kappa_s(u) = \kappa_s \cdot u$ . The main goal of this paper is to give an alternative proof of part (i) of this theorem without using the theory of singular connections. Instead, by studying immersed 2-spheres in manifolds  $X$  of simple type, we are able to show that the formal power series  $\mathbf{K}_X$  satisfies a system of homogeneous linear ordinary differential equations with constant coefficients and distinct characteristic roots. This is our key result. It follows that the formal power series  $\mathbf{K}_X$  is an analytic function, in fact a linear combination of exponentials. This proves (i).

Because we are working with immersed spheres rather than embedded surfaces, our version of part (ii) of the structure theorem is slightly different. It turns out that the characteristic roots of the system of ordinary differential equations which are satisfied by  $\mathbf{K}_X$  are geometrically significant, and as a corollary to their computation we have the following version of (ii):

**Theorem 1.1.** *Let  $X$  be a simply connected 4-manifold of simple type and let  $\{\kappa_s\}$  be the set of basic classes as above. If  $u \in H_2(X; \mathbf{Z})$  is represented by an immersed 2-sphere with  $p \geq 1$  positive double points, then for each  $s$*

$$(1) \quad 2p - 2 \geq u^2 + |\kappa_s \cdot u|.$$

**Theorem 1.2.** *Let  $X$  be a simply connected 4-manifold of simple type with basic classes  $\{\kappa_s\}$  as above. If the nontrivial class  $u \in H_2(X; \mathbf{Z})$  is represented by an immersed 2-sphere with no positive double points, and*

$$\{\kappa_s \mid s = 1, \dots, 2m\}$$

*is the collection of basic classes which violate the inequality (1), then  $\kappa_s \cdot u = \pm u^2$  for each such  $\kappa_s$ . Order these classes so that  $\kappa_s \cdot u =$*

$-u^2 (> 0)$  for  $s = 1, \dots, m$ . Then

$$\sum_{s=1}^m a_s e^{\kappa_s + u} - (-1)^{(1+b_X^+)/2} \sum_{s=1}^m a_s e^{-\kappa_s - u} = 0.$$

The above theorems are at the same time weaker and stronger than part (ii) of Kronheimer and Mrowka's theorem. Because one can always desingularize a double point, (ii) is stronger than the above theorems for classes of nonnegative self-intersection, where Kronheimer and Mrowka's theorem applies. However our theorems apply as well to classes of negative self-intersection.

Because we have avoided the use of singular connections, our proof of the structure theorem is reasonably short. This paper is not self-contained in that it uses techniques of C. Taubes in determining Mayer-Vietoris formulas for the Donaldson invariant when splitting off the neighborhood of an embedded 2-sphere. (See §4.) A relatively short proof of these formulas from first principles is provided by the thesis of W. Wieczorek [39]. Combined with this paper, it gives a rapid approach to the structure theorem.

As we have stated, our method is to study the effect on the Donaldson series of an embedded sphere with negative self-intersection. To parlay this into an understanding of the effect of an arbitrary immersed sphere we need to understand how blowing up a 4-manifold, i.e., forming the connected sum  $X \# \overline{\mathbf{CP}}^2$ , alters the Donaldson series. We have given such a formula in [14]. Under the simple type assumption, our result states that for  $c \in H_2(X; \mathbf{Z})$ ,  $\mathbf{K}_{X \# \overline{\mathbf{CP}}^2, c} = \mathbf{K}_{X, c} \cosh(e)$  where  $e \in H_2(\overline{\mathbf{CP}}^2; \mathbf{Z})$  is (the dual with respect to the intersection form of) the exceptional class. Further  $\mathbf{K}_{X \# \overline{\mathbf{CP}}^2, c+e} = -\mathbf{K}_{X, c} \sinh(e)$ . It is then easy to compute the Donaldson series for what turn out to be important 4-manifolds for our theory, the simply connected elliptic surfaces without multiple fibers and with Euler characteristic  $12n$ , denoted  $E(n)$ . In particular we show that

$$\mathbf{K}_{E(n)} = \sinh^{n-2}(f),$$

where  $f \in H_2(E(n))$  is the homology class of a generic fiber. There are corresponding computations for the  $\text{SO}(3)$  Donaldson series.

With these important computations under our belt, we begin the gauge theory which will evolve into showing that the formal power series  $\mathbf{K}_X$  satisfies a system of homogeneous linear ordinary differential equations with constant coefficients. At bottom, we show that the Donaldson invariant for arbitrary smooth 4-manifolds, when evaluated on products of powers of a homology class  $\alpha \in H_2(X)$  which is represented

by an embedded sphere and arbitrary  $z \in \mathbf{A}(\alpha^\perp)$ , satisfies specific relations with coefficients that only depend upon the self-intersection of the homology class. It is here that we rely upon recent work of C. Taubes ([35],[36],[37],[38]) to give us techniques for calculating Donaldson invariants in the presence of reducible connections. As we have pointed out above, we are able to avoid the complete generality of Taubes' theory because we are splitting along lens spaces.

Our main results concerning the structure of the Donaldson invariants for manifolds with simple type are a formal consequence of these fundamental relations. First, we translate these relations into relations among the derivatives of  $\mathbf{K}_X$  with respect to  $\alpha$  when evaluated on classes orthogonal to  $\alpha$ . We then use the blowup formula to extend these differential equations to all of  $H_2(X)$ . The universality of the coefficients in our fundamental relations implies that coefficients of these equations are constant. Utilizing the specific computations for the elliptic surfaces and their blowups, it is then an easy task to compute characteristic roots of our differential equations.

The next step is to choose a basis for the homology of  $X$ , represent this basis by immersed 2-spheres, and then blow up  $X$  until all these immersed spheres are represented by embedded spheres. The homology class of the immersed sphere changes under this operation, but the change depends only on the number of positive double points of its immersion. Using the blowup formula we see that  $\mathbf{K}_X$  satisfies a system of constant coefficient homogeneous linear ordinary differential equations whose coefficients now only depend upon the self-intersection of the basis elements and the number of positive double points in their immersed representatives. The main results follow from this.

This gives a rather complete qualitative description of the Donaldson series as well as strong lower bounds for the number of positive double points for immersed representatives of 2-dimensional homology classes under the assumption of simple type. Our blowup formula [14] and fundamental relations for embedded spheres (Theorem 4.8) are proved without the assumption of simple type, and the formal aspects of the argument to determine the existence of the basic classes can be extended to give results concerning arbitrary simply connected smooth 4-manifolds. However, it could very well be that any simply connected smooth 4-manifold with  $b^+ > 1$  has simple type.

## 2. The Donaldson invariant

In this section we outline the definition of the Donaldson invariant.

We refer the reader to [8] and [11] for a more complete treatment. Given an oriented simply connected 4-manifold with a generic Riemannian metric and an  $SU(2)$  or  $SO(3)$  bundle  $P$  over  $X$ , the moduli space of gauge equivalence classes of anti-self-dual connections on  $P$  is a manifold  $\mathcal{M}_X(P)$  whose dimension is

$$8c_2(P) - 3(1 + b_X^+)$$

if  $P$  is an  $SU(2)$  bundle, and is

$$-2p_1(P) - 3(1 + b_X^+)$$

if  $P$  is an  $SO(3)$  bundle. It will often be convenient to treat these two cases together by identifying  $\mathcal{M}_X(P)$  and  $\mathcal{M}_X(\text{ad}(P))$  for an  $SU(2)$  bundle  $P$ . Over the product  $\mathcal{M}_X(P) \times X$  there is a universal  $SO(3)$  bundle  $\mathbf{P}$  and there results a homomorphism  $\mu : H_i(X) \rightarrow H^{4-i}(\mathcal{M}_X(P))$  obtained by decomposing the class  $-\frac{1}{4}p_1(\mathbf{P}) \in H^4(\mathcal{M}_X \times X)$ . (Homology is always taken to have real coefficients unless it is otherwise adorned.) The basic idea of Donaldson's theory is that one should evaluate cup products of classes in the image of  $\mu$  against the fundamental class of  $\mathcal{M}_X(P)$ . To do this, one first needs to orient  $\mathcal{M}_X(P)$ . This is accomplished by orienting  $H_+^2(X)$  (see [9]). If  $P$  is an  $SO(3)$  bundle, we fix an integral lift of  $w_2(P) \in H^2(X; \mathbf{Z}_2)$  and always identify such a lift with its Poincaré dual  $c \in H_2(X; \mathbf{Z})$ . The Pontryagin number  $p_1(P)$  is congruent to  $c^2 \pmod{4}$ . If  $c$  and  $c'$  are two integral "lifts" of  $w_2(P)$ , then the difference in induced orientations is given by  $(-1)^{\binom{c-c'}{2}}$ . We say that  $c$  and  $c'$  are equivalent if they are congruent  $\pmod{2}$  and  $(-1)^{\binom{c-c'}{2}} = +1$ . The combination of the orientations of  $X$  and  $H_+^2(X)$  together with an equivalence class  $c$  of lifts of  $w_2(P)$  is called a "homology orientation" of  $X$ . (In case  $P$  is an  $SU(2)$  bundle, one chooses  $c = 0$ .) For a Kähler surface  $X$  with Kähler class  $K_X$ , there is a natural orientation induced from the Kähler structure and a choice of a lift  $c$  gives an orientation which differs from this one by  $(-1)^{\frac{1}{2}(c^2 + c \cdot K_X)}$  [9].

The moduli space  $\mathcal{M}_X(P)$  is, in general, noncompact and needs to be compactified before a fundamental class can be defined. The Uhlenbeck compactification  $\overline{\mathcal{M}}_X(P)$  is well-suited to this. However, this compactification is a stratified space and is not usually a manifold. Thus, to define a fundamental class one needs to insure that the singular set has codimension at least 2. This turns out to be the case where either  $w_2(P) \neq 0$  or  $w_2(P) = 0$ ,  $d > \frac{3}{4}(1 + b_X^+)$ . In practice, one is able to get around this latter restriction by blowing up  $X$  and considering bundles over  $X \# \overline{\mathbf{CP}}^2$  which are nontrivial when restricted to the exceptional divisor [29]. In [19] it is shown that for  $\alpha \in H_2(X; \mathbf{Z})$  the

classes  $\mu(\alpha) \in H^2(\mathcal{M}_X(P))$  extend over  $\overline{\mathcal{M}}_X(P)$ . When  $b_X^+$  is odd,  $\dim \mathcal{M}_X(P)$  is even, say equal to  $2d$ . In fact, a class  $c \in H_2(X; \mathbf{Z})$  and a nonnegative integer  $d \equiv -c^2 + \frac{1}{2}(1 + b^+) \pmod{4}$  determine an  $SO(3)$  bundle  $P_{c,d}$  over  $X$  with  $w_2(P_{c,d}) \equiv c \pmod{2}$  and formal dimension  $\dim \mathcal{M}_X(P_{c,d}) = 2d$ . For  $\bar{\alpha} = (\alpha_1, \dots, \alpha_d) \in H_2(X; \mathbf{Z})^d$ , write  $\mu(\bar{\alpha}) = \mu(\alpha_1) \cup \dots \cup \mu(\alpha_d)$ . Then one has

$$\langle \mu(\bar{\alpha}), [\overline{\mathcal{M}}_X(P_{c,d})] \rangle = \int_{\overline{\mathcal{M}}_X(P_{c,d})} \mu(\bar{\alpha})$$

when  $\mu(\bar{\alpha})$  is viewed as a  $2d$ -form.

If  $[1] \in H_0(X; \mathbf{Z})$  is the generator, then  $\nu = \mu([1]) = -\frac{1}{4}p_1(\beta) \in H^4(\mathcal{M}_X(P))$  where  $\beta$  is the basepoint fibration  $\tilde{\mathcal{M}}_X(P) \rightarrow \mathcal{M}_X(P)$  with  $\tilde{\mathcal{M}}_X(P)$  the manifold of anti-self-dual connections on  $P$  modulo based gauge transformations, i.e., those that are the identity on the fiber over a fixed basepoint. The class  $\nu$  extends over the Uhlenbeck compactification  $\overline{\mathcal{M}}_X(P)$  if  $w_2(P) \neq 0$ , and in case  $P$  is an  $SU(2)$  bundle, the class will extend under certain dimension restrictions. Once again, these restrictions can be done away with via the tricks mentioned above [29].

Consider the graded algebra

$$\mathbf{A}(X) = \text{Sym}_*(H_0(X) \oplus H_2(X))$$

where  $H_i(X)$  has degree  $\frac{1}{2}(4 - i)$ . The Donaldson invariant  $D_c = D_{X,c}$  is then an element of the dual algebra  $\mathbf{A}^*(X)$ , i.e., a linear function

$$D_c : \mathbf{A}(X) \rightarrow \mathbf{R}.$$

This is a homology orientation-preserving diffeomorphism invariant for manifolds  $X$  satisfying  $b_X^+ \geq 3$ . Throughout this paper we assume  $b_X^+ \geq 3$  and odd.

We let  $x \in H_0(X)$  be the generator [1] corresponding to the orientation. We shall reserve the use of  $1 \in \mathbf{A}(X)$  to denote the unit in degree 0. In case  $a + 2b = d > \frac{3}{4}(1 + b_X^+)$  and  $\alpha \in H_2(X)$ ,

$$D_c(\alpha^a x^b) = \langle \mu(\alpha)^a \nu^b, [\overline{\mathcal{M}}_X(P_{c,d})] \rangle.$$

It will be convenient to extend  $\mu$  over  $\mathbf{A}(X)$ , and write for  $z \in \mathbf{A}(X)$  of degree  $d$ ,  $D_c(z) = \langle \mu(z), [\overline{\mathcal{M}}_X(P_{c,d})] \rangle$ . Since such moduli spaces  $\mathcal{M}_X(P_{c,d})$  exist only for  $d \equiv -c^2 + \frac{1}{2}(1 + b_X^+) \pmod{4}$ , the Donaldson invariant  $D_c$  is defined only on elements of  $\mathbf{A}(X)$  whose total degree is congruent to  $-c^2 + \frac{1}{2}(1 + b_X^+) \pmod{4}$ . By definition,  $D_c$  is 0 on all elements of other degrees. We say that

$$\deg D_c \equiv -c^2 + \frac{1}{2}(1 + b_X^+) \pmod{4}.$$

When  $P$  is an  $SU(2)$  bundle one simply writes  $D$  or  $D_X$ .

If  $Y$  is a simply connected 4-manifold with boundary, one can similarly construct relative Donaldson invariants. A good reference for this is [28]. One formally works with bundles over the manifold  $Y \cup (\partial Y \times [0, \infty))$  with a cylindrical Riemannian metric on the end. Since the notation would become cumbersome if this were always denoted, we shall denote both  $Y$  and  $Y \cup (\partial Y \times [0, \infty))$  by “ $Y$ ”. Each based finite energy anti-self-dual connection  $A$  on a  $G = SO(3)$  or  $SU(2)$  bundle  $P$  over  $Y$  is asymptotically flat and has a well-defined boundary value  $\partial A \in \mathcal{R}_G(\partial Y)$ , the variety of  $G$ -representations of  $\pi_1(\partial Y)$ . (We identify based gauge equivalence classes of flat connections with representations.) We let  $\chi_G(\partial Y)$  denote the character variety  $\mathcal{R}_G(\partial Y)$  modulo conjugacy. In general, connections in cylindrical end moduli spaces need not decay exponentially to  $\partial A$  (cf. [28]). However, in this paper all such decay is exponential, since we shall be working with manifolds  $Y$  for which  $\chi_G(\partial Y)$  is a finite nondegenerate set  $\{\lambda_i\}$ . We shall denote by  $\tilde{\mathcal{M}}_Y(P)$  the moduli space of based finite-action anti-self-dual connections on  $P$  which decay exponentially to a flat connection on  $\partial Y$ . The map  $\partial : \tilde{\mathcal{M}}_Y(P) \rightarrow \mathcal{R}_G(\partial Y)$  is continuous and  $SO(3)$  equivariant [28], and so it induces a continuous map  $\partial : \mathcal{M}_Y(P) \rightarrow \chi_G(\partial Y)$ . Recalling that  $\chi_G(\partial Y)$  is a finite set  $\{\lambda_i\}$ , we denote by  $\mathcal{M}_Y(P)[\lambda_i]$  the union of the components of  $\mathcal{M}_Y(P)$  consisting of connections whose boundary value is in the conjugacy class  $\lambda_i$ . Again, there is a compactification of  $\mathcal{M}_Y(P)[\lambda_i]$ , the Uhlenbeck/Floer compactification, which carries a fundamental class. The classes  $\mu(\alpha)$  and  $\nu$  are defined as well in this cylindrical end situation, and in the obvious way we have the relative Donaldson invariants

$$D_{Y,c}[\lambda_i] : \mathbf{A}(Y) \rightarrow \mathbf{R}.$$

Note here that we are viewing  $c \in H_2(Y, \partial Y; \mathbf{Z})$ . Now suppose that  $X = Y_1 \cup Y_2$  with  $\partial Y_1 = -\partial Y_2$  a 3-manifold with, say  $\chi_G(\partial Y_i)$  a finite set  $\{\lambda_i\}$ . Consider a  $G$  bundle over  $X$  with an integral lift  $c \in H_2(X; \mathbf{Z})$  of  $w_2$ . Let  $c_i$  be integral lifts of  $w_2|_{Y_i}$ . Assume that  $H_2(\partial Y_i; \mathbf{Z}) = 0$ . Then

$$H_2(X; \mathbf{Z}) \rightarrow H_2(Y_1, \partial; \mathbf{Z}) \oplus H_2(Y_2, \partial; \mathbf{Z})$$

is injective, and we can unambiguously write  $c = c_1 + c_2$ . If  $u_i \in \mathbf{A}(Y_i)$  for  $i = 1, 2$  then

$$D_{X,c}(u_1, u_2) = \langle D_{Y_1, c_1}, D_{Y_2, c_2} \rangle(u_1, u_2) = \sum_{\lambda_i} D_{Y_1, c_1}[\lambda_i](u_1) \cdot D_{Y_2, c_2}[\lambda_i](u_2).$$

Generally, if  $\partial Y$  is a homology sphere, even though  $\chi_G(\partial Y)$  may not be discrete, one obtains formulas as above through the use of Floer



homology  $HF_*(\partial Y)$  [16], [1], [10]. In this case one has relative invariants

$$D_{Y,c} : \mathbf{A}(Y) \rightarrow HF_*(\partial Y).$$

It is then a theorem of Donaldson that if  $b_{Y_i}^+ > 0$  for both  $i$ ,

$$D_{X,c}(u_1, u_2) = \langle D_{Y_1,c_1}(u_1), D_{Y_2,c_2}(u_2) \rangle,$$

where the pairing is the Kronecker pairing of  $HF_j(\partial Y_1)$  with  $HF_j(-\partial Y_2) = HF_{-3-j}(\partial Y_1)$ .

Following [24], one considers the invariant

$$\hat{D}_{X,c} : \text{Sym}_*(H_2(X)) \rightarrow \mathbf{R}$$

defined by  $\hat{D}_{X,c}(u) = D_{X,c}((1 + \frac{x}{2})u)$ . Whereas  $D_{X,c}$  can be nonzero only in degrees congruent to  $-c^2 + \frac{1}{2}(1 + b^+) \pmod{4}$ ,  $\hat{D}_{X,c}$  can be nonzero in degrees congruent to  $-c^2 + \frac{1}{2}(1 + b^+) \pmod{2}$ . The *Donaldson series*  $\mathbf{D}_c = \mathbf{D}_{X,c}$  is defined by

$$\mathbf{D}_{X,c}(\alpha) = \hat{D}_{X,c}(\exp(\alpha)) = \sum_{d=0}^{\infty} \frac{\hat{D}_{X,c}(\alpha^d)}{d!}$$

for all  $\alpha \in H_2(X)$ . This is a formal power series on  $H_2(X)$ .

A simply connected 4-manifold  $X$  is said to have *simple type* if the relation  $D_{X,c}(x^2 z) = 4 D_{X,c}(z)$  is satisfied by its Donaldson invariant for all  $z \in \mathbf{A}(X)$  and all  $c \in H_2(X; \mathbf{Z})$ . This important definition is due to Kronheimer and Mrowka [24] and was observed to hold for many 4-manifolds. In terms of  $\hat{D}_{X,c}$ , the simple type condition is that  $\hat{D}_{X,c}(zx) = 2\hat{D}_{X,c}(z)$  for all  $z \in \mathbf{A}(X)$  and all  $c \in H_2(X; \mathbf{Z})$ . The assumption of simple type assures that for each  $c$ , the complete Donaldson invariant  $D_{X,c}$  is determined by the Donaldson series  $\mathbf{D}_{X,c}$ . Further, as we observed in [14], the simple type condition naturally arises when certain Weierstrass elliptic functions associated with the Donaldson invariant of  $X \# \overline{\mathbf{C}\mathbf{P}}^2$  degenerate. It is an open question whether all 4-manifolds are of simple type. As we shall see (Corollary 3.11), all manifolds which contain a copy of the Milnor fiber  $B(2, 3, 7)$  of the  $(2, 3, 7)$  Brieskorn singularity have simple type. This includes, for example, all simply connected elliptic surfaces with  $p_g \geq 1$ . We shall show in Theorem 5.14 that  $X$  has simple type provided that the condition  $D_X(zx^2)$  is satisfied for its  $SU(2)$  invariant (with no a priori conditions placed on other invariants  $D_{X,c}$ ). This result has also been proved independently by Kronheimer and Mrowka.

We conclude this section with some conventions regarding symmetric functions. Two important symmetric functions are the degree  $d$  homogeneous part of the Donaldson invariant,  $D_{X,c}^{(d)} \in \text{Sym}^d(H_2(X))$ ,

and the intersection form of  $X$ ,  $Q \in \text{Sym}^2(H_2(X))$ . Further, for each  $\alpha \in H_2(X; \mathbf{Z})$  there is the dual form  $\tilde{\alpha} \in \text{Sym}^1(H_2(X; \mathbf{Z}))$  defined by  $\tilde{\alpha}(\beta) = \alpha \cdot \beta$ . We will usually drop the “tilde” and identify each  $\alpha$  with its dual. Of course, we can also identify  $\tilde{\alpha}$  with the Poincaré dual of  $\alpha$ , i.e., as an element of  $H^2(X)$ . Beware that if  $\alpha \in H_2(X)$ , there is the possible, but unlikely, opportunity for confusion between use of the same notation for the degree 2 element  $\alpha^2 \in \mathbf{A}(X)$  and the intersection number  $\alpha^2 = \alpha \cdot \alpha$ . If  $\varphi_i \in \text{Sym}^{d_i}(H_2(X))$  for  $i = 1, 2$ , then the product  $\varphi_1\varphi_2 \in \text{Sym}^{d_1+d_2}(H_2(X))$  is defined by

$$\varphi_1\varphi_2(\alpha_1, \dots, \alpha_{d_1+d_2}) = \frac{1}{(d_1 + d_2)!} \sum_{\sigma \in S_{d_1+d_2}} \varphi_1(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(d_1)}) \varphi_2(\alpha_{\sigma(d_1+1)}, \dots, \alpha_{\sigma(d_1+d_2)}).$$

where  $S_{d_1+d_2}$  is the symmetric group on  $d_1 + d_2$  letters.

### 3. Elliptic surfaces without multiple fibers

The smooth simply connected elliptic surfaces  $E(n)$  without multiple fibers are classified up to diffeomorphism by their holomorphic Euler characteristic,  $\chi(E(n)) = n$ , or alternatively by their Euler number  $12n$ . With this notation, the  $K3$ -surface is  $E(2)$ . The Donaldson invariants for  $E(n)$ ,  $n \geq 2$ , are important for our theory. In this section we show how the blowup formula [14] and a particular inductive construction of  $E(n)$  from  $E(n - 1)$  allow the full computation of their Donaldson invariants. In particular we show that each  $E(n)$ ,  $n \geq 2$ , has simple type and

**Theorem 3.1.** *The Donaldson series of the elliptic surfaces  $E(n)$  are given by*

$$\begin{aligned} \mathbf{D}_{E(n),c} &= (-1)^{\frac{1}{2}(c^2+(n-2)c \cdot f)} \exp(Q/2) \sinh^{n-2}(f) \text{ if } c \cdot f \equiv 0 \pmod{2}, \\ \mathbf{D}_{E(n),c} &= (-1)^{\frac{1}{2}(c^2+(n-2)c \cdot f)} \exp(Q/2) \cosh^{n-2}(f) \text{ if } c \cdot f \equiv 1 \pmod{2}, \end{aligned}$$

where  $f \in H_2(E(n); \mathbf{Z})$  is the homology class of the fiber.

#### 3.1. Manifolds split by $\Sigma(2, 3, 11)$

Let  $X$  be an oriented simply connected 4-manifold with  $b^+$  odd and  $\geq 3$ . Suppose that  $X = X_1 \cup X_2$  with  $\partial X_1 = -\partial X_2 = \Sigma$  an integral homology 3-sphere. As we mentioned in the last section, it is a theorem of Donaldson that if both  $X_1$  and  $X_2$  have  $b^+ > 0$ , then the Donaldson invariants of  $X$  may be expressed as a pairing

$$(2) \quad D_{X,c} = \langle D_{X_1,c_1}, D_{X_2,c_2} \rangle$$

where the  $D_{X_i, c_i}$  are relative Donaldson invariants, taking their values in the Floer homology of  $\Sigma$  (and  $c = c_1 + c_2$ ). Now assume in addition that  $X$  has simple type and that  $\Sigma = \Sigma(2, 3, 11)$ , the  $(2,3,11)$ -Brieskorn homology 3-sphere. We shall see in §3.4 that the simply connected elliptic surfaces with  $p_g \geq 1$  can be split in this way. The fact that is needed about  $\Sigma(2, 3, 11)$  is that its Floer homology is uncomplicated, as is shown in [12].

**Lemma 3.2.** *The Floer homology  $HF_*(\Sigma(2, 3, 11))$  is a copy of  $\mathbf{Z}$  in odd dimensions and vanishes in even dimensions.*

Let  $\lambda_j$  denote the generator of  $HF_*(\Sigma(2, 3, 11))$  in dimension  $j$ , and let  $D_{Y_i, c_i}[\lambda_j]$  denote the relative invariants in  $HF_j(\Sigma(2, 3, 11))$ . Let  $c = c_1 + c_2 \in H_2(X_1; \mathbf{Z}) \oplus H_2(X_2; \mathbf{Z}) \cong H_2(X; \mathbf{Z})$ , and let  $u_i \in \mathbf{A}(X_i)$  have degree  $d_i$ . Suppose that  $d_1 + d_2 \equiv \deg D_{X, c} \pmod{2}$ . Let

$$j_i = -2d_i - 2c_i^2 - 3(1 + b_{X_i}^+) \pmod{8}.$$

Then  $j_1 \equiv -3 - j_2 \pmod{8}$ ; so, letting  $j = j_1$  we have

$$\begin{aligned} \langle D_{X_1, c_1}((1 + \frac{x}{2})u_1), D_{X_2, c_2}((1 + \frac{x}{2})u_2) \rangle \\ = \langle D_{X_1, c_1}[\lambda_j](u_1) + D_{X_1, c_1}[\lambda_{j+4}](u_1 \frac{x}{2}), \\ D_{X_2, c_2}[\lambda_j](u_2) + D_{X_2, c_2}[\lambda_{j+4}](u_2 \frac{x}{2}) \rangle \\ = 2\hat{D}_{X, c}(u_1 u_2), \end{aligned}$$

since  $X$  has simple type.

To correct for this factor of 2, we view the Floer homology of  $\Sigma(2, 3, 11)$  as  $\mathbf{Z}_4$ -graded. The generators of the Floer homology are then  $\alpha$  in dimension  $1 \pmod{4}$  and  $\beta$  in dimension  $3 \pmod{4}$ . So  $\alpha$  corresponds to  $\lambda_1$  and  $\lambda_5$  and  $\beta$  to  $\lambda_3$  and  $\lambda_7$ . If  $j_1 \equiv -3 - j_2 \pmod{4}$ , then for  $\gamma = \alpha$  or  $\beta$  we have

$$\begin{aligned} \langle D_{X_1, c_1}[\gamma](u_1), D_{X_2, c_2}[\gamma](u_2) \rangle &= \langle D_{X_1, c_1}[\lambda_{j_1}](u_1), D_{X_2, c_2}[\lambda_{j_2}](u_2) \rangle \\ &= D_{X, c}(u_1 u_2); \end{aligned}$$

for  $j_1 \equiv -3 - j_2 \pmod{8}$ , and

$$\begin{aligned} \langle D_{X_1, c_1}[\gamma](u_1), D_{X_2, c_2}[\gamma](u_2) \rangle &= \langle D_{X_1, c_1}[\lambda_{j_1+4}](u_1 \frac{x}{2}), D_{X_2, c_2}[\lambda_{j_2}](u_2) \rangle \\ &= \langle D_{X_1, c_1}[\lambda_{j_1}](u_1), D_{X_2, c_2}[\lambda_{j_2+4}](u_2 \frac{x}{2}) \rangle \\ &= \hat{D}_{X, c}(u_1 u_2). \end{aligned}$$

for  $j_1 + 4 \equiv -3 - j_2 \pmod{8}$ .

In other words, if  $j_1 \equiv -3 - j_2 \equiv 1 \pmod{4}$  then

$$\hat{D}_{X,c}(u_1 u_2) = \langle D_{X_1,c_1}[\alpha](u_1), D_{X_2,c_2}[\alpha](u_2) \rangle,$$

and if  $j_1 \equiv -3 - j_2 \equiv 3 \pmod{4}$  then

$$\hat{D}_{X,c}(u_1 u_2) = \langle D_{X_1,c_1}[\beta](u_1), D_{X_2,c_2}[\beta](u_2) \rangle.$$

**Lemma 3.3.** *The Donaldson series of the above  $X$  is given by*

$$\mathbf{D}_{X,c} = \langle \mathbf{D}_{X_1,c_1}[\alpha], \mathbf{D}_{X_2,c_2}[\alpha] \rangle + \langle \mathbf{D}_{X_1,c_1}[\beta], \mathbf{D}_{X_2,c_2}[\beta] \rangle.$$

*The pairing of these relative Donaldson series is multiplication of the corresponding formal power series.*

Next suppose that as before,  $X$  has simple type and that  $X = X_1 \cup X_2$  with  $\partial X_1 = -\partial X_2 = \Sigma(2, 3, 11)$ , but now assume that  $b_{X_1}^+ = 0$  and  $b_{X_2}^+ > 0$ . The Mayer-Vietoris argument that gives Donaldson's theorem (2) is now complicated by the fact that there may be reducible connections on  $X_1$ . Each of these will have as boundary value the trivial flat connection  $\vartheta$  over  $\Sigma(2, 3, 11)$ . However, in the next lemma we see that these reducible connections cause problems in only half the possible cases.

**Lemma 3.4.** *Suppose that  $X = X_1 \cup X_2$  with  $\partial X_1 = -\partial X_2 = \Sigma(2, 3, 11)$ , and assume that  $b_{X_1}^+ = 0$  and  $b_{X_2}^+ > 0$ . Let  $c_i \in H_2(X_i; \mathbf{Z})$ ,  $i = 1, 2$ . If  $u_i$  has degree  $d_i$  in  $\mathbf{A}(X_i)$  and  $d_1 \not\equiv c_1^2 \pmod{2}$ , then*

$$\hat{D}_{X,c_1+c_2}(u_1 u_2) = \langle D_{X_1,c_1}[\beta](u_1), D_{X_2,c_2}[\beta](u_2) \rangle.$$

*Proof.* The proof will use techniques which are discussed at more length in §4. These techniques show that contributions to  $D_{X,c_1+c_2}(u_1 u_2)$  arise from products of based moduli spaces  $\tilde{\mathcal{M}}_{X_1}[\zeta] \times \tilde{\mathcal{M}}_{X_2}[\zeta]$ , divided out by the diagonal  $SO(3)$ , where  $\zeta$  denotes  $\alpha, \beta$ , or  $\vartheta$ . We are concerned with the case  $\zeta = \vartheta$ . Since the stabilizer of  $\vartheta$  is 3 ( $> 0$ )-dimensional, a counting argument shows that the divisor corresponding to  $u_1$  does not intersect  $\mathcal{M}_{X_1}[\vartheta]$  transversely and that the based moduli space  $\tilde{\mathcal{M}}_{X_1}[\vartheta]$  contains reducible nontrivial orbits as weak limits. Each of these reducible orbits is a 2-sphere. The class  $\tilde{\mu}(u_1)$  extends to an  $SO(3)$  equivariant cohomology class  $\hat{\mu}(u_1) \in H_{SO(3)}^{2d_1}(\mathcal{M}_{X_1}[\vartheta]; \mathbf{R})$  (see [37] and §4). Since each (based) connection in  $\tilde{\mathcal{M}}_{X_1}[\vartheta]$  has the same asymptotic value, which we can identify with  $1 \in SO(3)$ , there is an  $SO(3)$ -equivariant push-forward map

$$\partial_* : H_{SO(3)}^*(\tilde{\mathcal{M}}_{X_1}[\vartheta]; \mathbf{R}) \rightarrow H_{SO(3)}^*(\{1\}; \mathbf{R}).$$

This push-forward map is not given by a standard construction because the fibers of  $\partial$  are not compact. The necessary construction is given by Taubes [38]. The fiber dimension of the map  $\partial : \tilde{\mathcal{M}}_{X_1} \rightarrow SO(3)$  is equal to  $\dim \mathcal{M}_{X_1}[\vartheta] \equiv -2c_1^2 - 8k$  for some  $k$ . There is similarly a pullback map

$$\partial^* : H_{SO(3)}^*(\{1\}; \mathbf{R}) \rightarrow H_{SO(3)}^*(\tilde{\mathcal{M}}_{X_2}[\vartheta]; \mathbf{R}).$$

Furthermore, since  $SO(3)$  acts freely on  $\tilde{\mathcal{M}}_{X_2}[\vartheta]$ , there is an isomorphism

$$\pi^* : H^*(\mathcal{M}_{X_2}[\vartheta]; \mathbf{R}) \cong H_{SO(3)}^*(\tilde{\mathcal{M}}_{X_2}[\vartheta]; \mathbf{R}).$$

Then the corresponding contribution to the Donaldson invariant is obtained from the pairing  $\langle \pi^* \partial^* \partial_* \tilde{\mu}(u_1) \cup \mu(u_2), [\mathcal{M}_{X_2}[\vartheta]] \rangle$ , which is defined by integration. (Again, see [35], [36], [37], [4] and §4 for this.) However, under the hypothesis of the lemma,  $2d_1 + 2c_1^2 \equiv 2 \pmod{4}$ ; so

$$\partial_* \tilde{\mu}(u_1) \in H_{SO(3)}^{2d_1+2c_1^2+8k}(\{1\}; \mathbf{R}) \cong H^{2d_1+2c_1^2+8k}(BSO(3); \mathbf{R}) = 0.$$

This means that the boundary value  $\vartheta$  does not contribute to the calculation of the invariant  $D_{X, c_1+c_2}(u_1 u_2)$ . The correct boundary value lives in  $HF_j(\Sigma(2, 3, 11))$ , where

$$j \equiv -(2c_1^2 + 3 + 2d_1) \equiv -(2 + 3) \pmod{4}.$$

Thus it must be  $\beta$ .

### 3.2. Embedded $-2$ -spheres and Donaldson invariants

We next study 4-manifolds which have a homology class  $\sigma$  represented by an embedded 2-sphere  $S$  of self-intersection  $\sigma^2 = -2$ . We shall see that such a class has a profound effect on the Donaldson invariants of  $X$ . Let  $\langle \sigma \rangle^\perp$  denote  $\{\alpha \in H_2(X) \mid \sigma \cdot \alpha = 0\}$  and let

$$\mathbf{A}(\sigma^\perp) = \mathbf{A}_X(\sigma^\perp) = \text{Sym}_*(H_0(X) \oplus \langle \sigma \rangle^\perp).$$

We begin by reviewing two basic relations proved in [14]. The first is due to D. Ruberman.

**Theorem 3.5** (Ruberman [32]). *Suppose that  $\sigma \in H_2(X; \mathbf{Z})$  with  $\sigma^2 = -2$  is represented by an embedded sphere. Let  $c \in H_2(X; \mathbf{Z})$  satisfy  $c \cdot \sigma \equiv 0 \pmod{2}$ . Then for all  $z \in \mathbf{A}(\sigma^\perp)$  we have*

$$D_c(\sigma^2 z) = 2 D_{c+\sigma}(z).$$

We also need a formula for spheres of square  $-3$ .

**Theorem 3.6.** [14] *Suppose that  $\sigma \in H_2(X; \mathbf{Z})$  is represented by an embedded 2-sphere with self-intersection  $-3$ . Let  $c \in H_2(X; \mathbf{Z})$  satisfy  $c \cdot \sigma \equiv 0 \pmod{2}$ . Then for all  $z \in \mathbf{A}(\sigma^\perp)$  we have*

$$D_c(\sigma z) = -D_{c+\sigma}(z).$$

In [14] we also showed

**Proposition 3.7.** [14] *Suppose that  $\sigma \in H_2(X; \mathbf{Z})$  is represented by an embedded 2-sphere with self-intersection  $-2$ , and let  $c \in H_2(X; \mathbf{Z})$  with  $c \cdot \sigma \equiv 0 \pmod{2}$ . Then for all  $z \in \mathbf{A}(\sigma^\perp)$*

$$D_c(\sigma^4 z) = -4 D_c(\sigma^2 xz) - 4 D_c(z).$$

For a 4-manifold  $X$  and a class  $\kappa \in H_2(X; \mathbf{Z})$ , let  $\text{Diff}_\kappa(X)$  be the group of orientation-preserving diffeomorphisms  $f$  of  $X$  which satisfy  $f_*(\kappa) = \kappa$ . Also, let  $\text{Aut}(X)$  be the group of automorphisms of  $H_2(X; \mathbf{Z})$ , which preserve the intersection form  $Q$ . Then  $X$  is said to have a *big diffeomorphism group* with respect to  $\kappa$  if the image of  $\text{Diff}_\kappa(X)$  in  $\text{Aut}(X)$  has finite index. For example, the simply connected minimal elliptic surfaces with  $p_g \geq 1$  have a big diffeomorphism group with respect to their canonical class [19]. From the assumption of big diffeomorphism group with respect to  $\kappa$  it follows that for each  $d$ , the degree  $d$  homogeneous part  $D_{X,c}^{(d)}$  of the Donaldson invariants  $D_{X,c}$  and  $D_{X,c}(\frac{x}{2})$  (and hence  $\hat{D}_{X,c}$ ) are polynomials in the intersection form  $Q$  and the class  $\kappa$  when viewed as linear maps  $\text{Sym}_*(H_2(X)) \rightarrow \mathbf{R}$ . If  $\frac{1}{2}(1 + b_X^+) \equiv 0 \pmod{2}$ , we can then write

$$\begin{aligned} \frac{1}{(2d)!} \hat{D}_X^{(2d)} &= c_0 \frac{Q^d}{2^d d!} + c_2 \frac{Q^{d-1} \kappa^2}{2^{d-1} (d-1)! 2!} + \cdots + c_{2t} \frac{Q^{d-t} \kappa^{2t}}{2^{d-t} (d-t)! (2t)!} \\ &\quad + \cdots + c_{2d} \frac{\kappa^{2d}}{(2d)!}, \end{aligned}$$

and if  $\frac{1}{2}(1 + b_X^+) \equiv 1 \pmod{2}$  we can write

$$\begin{aligned} \frac{1}{(2d+1)!} \hat{D}_X^{(2d+1)} &= c_1 \frac{Q^d}{2^d d!} \kappa + \cdots + c_{2t+1} \frac{Q^{d-t} \kappa^{2t+1}}{2^{d-t} (d-t)! (2t+1)!} \\ &\quad + \cdots + c_{2d+1} \frac{\kappa^{2d+1}}{(2d+1)!}. \end{aligned}$$

The crucial fact here is that if  $X$  contains an embedded 2-sphere of self-intersection  $-2$  which is orthogonal to  $\kappa$ , the coefficients  $c_j$  are independent of the homogeneous degree. Related results were first observed by Peter Kronheimer (unpublished).

**Proposition 3.8.** *Suppose that  $\sigma \in H_2(X; \mathbf{Z})$  is represented by an embedded 2-sphere, and suppose also that  $\sigma^2 = -2$  and  $\sigma \cdot \kappa = 0$ . Then the coefficients  $c_j$  above are independent of the homogeneous degree  $d$  of  $\hat{D}_X^{(d)}$ . Similarly, if we express  $\hat{D}_{X,\omega}$  as a polynomial in  $Q$  and  $\kappa$  with coefficients  $c'_j$ , and if also  $\omega \cdot \sigma \equiv 0 \pmod{2}$ , then the  $c'_j$  are independent of the degree  $d$  of  $\hat{D}_{X,\omega}^{(d)}$ .*

*Proof.* To fix notation, assume that  $\frac{1}{2}(1 + b_X^+) \equiv 0 \pmod{4}$ . Write

$$\frac{1}{(2d - 2)!} \hat{D}_{X,\sigma}^{(2d-2)} = c'_0 \frac{Q^{d-1}}{2^{d-1}(d-1)!} + c'_2 \frac{Q^{d-2} \kappa^2}{2^{d-2}(d-2)! 2!} + \dots$$

Consider an  $\alpha \in H_2(X)$  which satisfies  $\alpha \cdot \sigma = \alpha \cdot \kappa = 0$ . Then

$$\begin{aligned} \hat{D}_X(\alpha^{2d-2}\sigma^2) &= (2d)!c_0 \frac{Q^d}{2^d d!} (\alpha^{2d-2}\sigma^2) \\ &= \frac{(2d)!c_0 d \cdot 2!(2d-2)!}{2^d d! (2d)!} Q^{d-1}(\alpha^{2d-2})Q(\sigma^2) \\ &= \frac{-2c_0(2d-2)!}{2^{d-1}(d-1)!} Q^{d-1}(\alpha^{2d-2}). \end{aligned}$$

But, by Theorem 3.5 we have

$$\hat{D}_X(\alpha^{2d-2}\sigma^2) = 2 \hat{D}_{X,\sigma}(\alpha^{2d-2}) = 2(2d-2)! \frac{c'_0}{2^{d-1}(d-1)!} Q^{d-1}(\alpha_{2d-2}).$$

Thus  $c'_0 = -c_0$ .

Next take  $\beta \in H_2(X)$  with  $\beta^2 = 0$ ,  $\beta \cdot \kappa = 1$ , and  $\beta \cdot \sigma = \beta \cdot \alpha = 0$ . Then for  $j \geq 1$

$$\begin{aligned} \hat{D}_X(\alpha^{2d-2j-2}\beta^{2j}\sigma^2) &= (2d)! \frac{c_{2j}}{2^{d-j}(d-j)!} Q^{d-j} \cdot \frac{\kappa^{2j}}{2^j!} (\alpha^{2d-2j-2}\beta^{2j}\sigma^2) \\ &= -c_{2j} \frac{(2d-2j-2)!}{2^{d-j-2}(d-j-1)!} Q^{d-j-1}(\alpha^{2d-2j-2}) \\ &= 2 \hat{D}_{X,\sigma}(\alpha^{2d-2j}\beta^2) \\ &= c'_{2j} \frac{(2d-2j-2)!}{2^{d-j-2}(d-j-1)!} Q^{d-j-1}(\alpha^{2d-2j-2}). \end{aligned}$$

Thus  $c'_{2j} = -c_{2j}$  for all  $j \geq 0$ .

We now perform the same procedure on the  $SO(3)$ -Donaldson invariants  $D_{X,\sigma}$ . Write

$$\frac{1}{(2d-4)!} \hat{D}_X^{(2d-4)} = c''_0 \frac{Q^{d-2}}{2^{d-2}(d-2)!} + c''_2 \frac{Q^{d-3} \kappa^2}{2^{d-3}(d-3)! 2!} + \dots$$

Note that Theorem 3.5 implies that for all  $z \in \mathbf{A}(\sigma^\perp)$ ,

$$\hat{D}_{X,\sigma}(\sigma^2 z) = 2 \hat{D}_{X,2\sigma}(z) = 2(-1)^{\sigma^2} \hat{D}_X(z) = 2 \hat{D}_X(z).$$

Using the expansion of  $\hat{D}_X^{(2d-4)}$  and applying Theorem 3.5, we get

$$\begin{aligned} \hat{D}_{X,\sigma}(\alpha^{2d-4} \sigma^2) &= (2d-2)! \frac{c'_0}{2^{d-1}(d-1)!} Q^{d-1}(\alpha^{2d-4} \sigma^2) \\ &= -c'_0 \frac{(2d-4)!}{2^{d-3}(d-2)!} Q^{d-2}(\alpha^{2d-4}) = 2 \hat{D}_X(\alpha^{2d-4}) \\ &= 2(2d-4)! \frac{c''_0}{2^{d-2}(d-2)!} Q^{d-2}(\alpha^{2d-4}). \end{aligned}$$

Thus  $c'_0 = -c''_0$  and so  $c_0 = c''_0$ . In a similar fashion we obtain that  $c_{2j} = c''_{2j}$  for all  $j$ .

Finally consider  $\hat{D}_X^{(2d-2)} = D_X^{(2d)}(\frac{x}{2})$ . Because  $X$  has a big diffeomorphism group we can write

$$\frac{1}{(2d-2)!} \hat{D}_X^{(2d-2)} = \hat{c}_0 \frac{Q^{d-1}}{2^{d-1}(d-1)!} + \hat{c}_2 \frac{Q^{d-2}}{2^{d-2}(d-2)!} \frac{\kappa^2}{2!} + \dots$$

Proposition 3.7 implies that

$$D_X(\alpha^{2d} \sigma^4) = -8 D_X(\alpha^{2d} \sigma^2 \frac{x}{2}) - 4 D_X(\alpha^{2d}).$$

Expanding as above, and using the fact that  $c''_0 = c_0$ , we have  $\hat{c}_0 = c_0$ , and continuing as above,  $\hat{c}_i = c_i$  for all  $i$ . This completes the proof in the  $SU(2)$  case with  $\frac{1}{2}(1 + b^+_X) \equiv 0 \pmod{4}$ . A similar proof suffices when  $\frac{1}{2}(1 + b^+_X) \not\equiv 0 \pmod{4}$ , and the same proof also works in the  $SO(3)$  case.

We now show that such manifolds have simple type. This implies, for example, that all simply connected elliptic surfaces (with  $b^+ \geq 3$ ), complete intersections, and Moishezon [27] and Salvetti [33] surfaces have simple type.

**Theorem 3.9.** *Let  $X$  be a simply connected 4-manifold which has a big diffeomorphism group with respect to a class  $\kappa \in H_2(X; \mathbf{Z})$ , and let  $\omega \in H_2(X; \mathbf{Z})$ . Suppose that  $\sigma \in H_2(X; \mathbf{Z})$  is represented by an embedded 2-sphere of square  $-2$  such that  $\sigma \cdot \kappa = 0$ , and  $\sigma \cdot \omega \equiv 0 \pmod{2}$ . Then for all  $z \in \mathbf{A}(X)$ , we have  $D_{X,\omega}(zx^2) = 4D_{X,\omega}(z)$ . If  $\omega^2 + \frac{1}{2}(1 + b^+) \equiv 0 \pmod{2}$  then*

$$D_{X,\omega} = \exp(Q/2) \sum c_{2i,\omega} \frac{\kappa^{2i}}{(2i)!},$$



and if  $\omega^2 + \frac{1}{2}(1 + b^+) \equiv 1 \pmod{2}$  then

$$D_{X,\omega} = \exp(Q/2) \sum c_{2i+1,\omega} \frac{\kappa^{2i+1}}{(2i+1)!}.$$

*Proof.* For simplicity of notation, consider the  $SU(2)$  case  $\omega = 0$  with  $\frac{1}{2}(1 + b_X^+) \equiv 0 \pmod{4}$  as above. Then Proposition 3.8 shows that for each  $d \equiv \frac{1}{2}(1 + b^+) \pmod{4}$  the homogeneous invariants  $D_X^{(2d+4)}(\frac{x}{2}) = \hat{D}_X^{(2d+2)}$  and  $D_X^{(2d)}$  share the same coefficients in that

$$\begin{aligned} \frac{1}{2d!} D_X^{(2d)} &= \sum_{i=0}^d c_{2i} \frac{Q^{d-i}}{2^{d-i}(d-i)!} \frac{\kappa^{2i}}{(2i)!}, \\ \frac{1}{(2d+2)!} D_X^{(2d+4)}\left(\frac{x}{2}\right) &= \sum_{i=0}^{d+1} c_{2i} \frac{Q^{d+1-i}}{2^{d+1-i}(d+1-i)!} \frac{\kappa^{2i}}{(2i)!}. \end{aligned}$$

The technique of Proposition 3.8 also shows that

$$\frac{1}{(2d+4)!} D_X^{(2d+8)}\left(\frac{x^2}{4}\right) = \sum_{i=0}^{d+2} c_{2i} \frac{Q^{d+2-i}}{2^{d+2-i}(d+2-i)!} \frac{\kappa^{2i}}{(2i)!},$$

independent of degree. Thus also,

$$\frac{1}{2d!} D_X^{(2d+4)}\left(\frac{x^2}{4}\right) = \sum_{i=0}^d c_{2i} \frac{Q^{d-i}}{2^{d-i}(d-i)!} \frac{\kappa^{2i}}{(2i)!} = \frac{1}{(2d)!} D_X^{(2d)};$$

so  $D_{X,\omega}(zx^2) = 4D_{X,\omega}(z)$ . The other cases are similar.

**Corollary 3.10.** *The K3 surface,  $E(2)$ , has simple type, and*

$$D_{E(2),c} = (-1)^{\frac{c^2}{2}} \exp(Q/2).$$

*Proof.* The diffeomorphism group  $\text{Diff}(E(2))$  acts on  $H_2(E(2); \mathbf{Z}_2)$  with exactly 3 orbits, namely  $\{0\}$ ,  $\{\alpha \neq 0 | \alpha^2 \equiv 0 \pmod{4}\}$ , and  $\{\alpha | \alpha^2 \equiv 2 \pmod{4}\}$ . These are represented by 0,  $f$ , and  $s_2$ , the class of a section. Thus, keeping in mind the rule  $D_{c'} = (-1)^{\frac{c-c'}{2}} D_c$  for  $c \equiv c' \pmod{2}$ , we may assume that  $c$  is one of these 3 classes.

In case  $c = 0$  or  $s_2$ , we let  $\sigma = s_2$ . If  $c = f$ , then there is a different elliptic fibration whose section  $\sigma = s'_2$  satisfies  $s'_2 \cdot f = 0$ . Now  $E(2)$  has a big diffeomorphism group (with respect to  $\kappa = 0$ ); so we may apply Theorem 3.9 to see first that  $E(2)$  has simple type, and second, that  $D_{E(2),c} = a_c \exp(Q/2)$  for some constant  $a_c$ . For  $c^2 \equiv 2 \pmod{4}$  it is known that  $D_{E(2),c}(1) = -1$ . For  $c^2 \equiv 0 \pmod{4}$ , find a class  $\sigma$  orthogonal to  $c$  and represented by an embedded 2-sphere of square  $-2$ .

Then  $D_{E(2),c}(\sigma^2) = 2D_{E(2),c+\sigma}(1) = -2$  by Theorem 3.5. It follows that for any  $c$ ,  $a_c = (-1)^{\frac{1}{2}c^2}$ . Hence we obtain the equation for  $D_{E(2),c}$ .

Corollary 3.10 has been known for some time.

**Corollary 3.11.** *Any simply connected 4-manifold (with  $b^+ \geq 3$  and odd) which contains the Brieskorn manifold  $B(2, 3, 7)$  has simple type.*

*Proof.* Write  $X = B(2, 3, 7) \cup Y$  with  $B(2, 3, 7) \cap Y = \Sigma(2, 3, 7)$ . The Floer homology of  $\Sigma(2, 3, 7)$  is  $HF_i(\Sigma(2, 3, 7)) = \mathbf{Z}$  for  $i = 3, 7$  and is 0 in other dimensions [12]. Let  $\lambda_3$  and  $\lambda_7$  be generators. If  $\alpha_1 \in H_2(B(2, 3, 7))$  and  $\alpha_2 \in H_2(Y)$  and  $d_1 + d_2 = d \equiv -c^2 + \frac{1}{2}(1 + b_X^+) \pmod{4}$  and  $c = c_B + c_Y$ , then

$$(3) \quad D_{X,c}(\alpha_1^{d_1} \alpha_2^{d_2} x^2) = \langle D_{B(2,3,7),c_B}[\lambda_j](\alpha_1^{d_1} x^2), D_{Y,c_Y}[\lambda_j](\alpha_2^{d_2}) \rangle$$

for  $j = 3$  or  $7$ .

Similarly,  $E(2) = B(2, 3, 7) \cup E_{10}$ . Note that  $b_{E_{10}}^+ = 1$ . If  $j = 3$  set  $c_E = 0$ , and if  $j = 7$  let  $c_E$  be any class in  $H_2(E_{10}; \mathbf{Z})$  with square  $-2$ . Then since  $E(2)$  has simple type,

$$\begin{aligned} D_{E(2),c_B+c_E}(\alpha_1^{d_1} x^2) &= \langle D_{B(2,3,7),c_B}[\lambda_j](\alpha_1^{d_1} x^2), D_{E_{10},c_E}[\lambda_j](1) \rangle \\ &= 4D_{E(2),c_B+c_E}(\alpha_1^{d_1}) \\ &= \langle 4D_{B(2,3,7),c_B}[\lambda_j](\alpha_1^{d_1}), D_{E_{10},c_E}[\lambda_j](1) \rangle. \end{aligned}$$

But  $D_{E(2),c'}(1) = -1$  for any  $c'$  of square congruent to 2 (mod 4). Thus  $D_{E_{10},c_E}[\lambda_j](1) \neq 0$ ; so

$$D_{B(2,3,7),c_B}[\lambda_j](\alpha_1^{d_1} x^2) = 4D_{B(2,3,7),c_B}[\lambda_j](\alpha_1^{d_1}).$$

It then follows from (3) that

$$\begin{aligned} D_{X,c}(\alpha_1^{d_1} \alpha_2^{d_2} x^2) &= \langle 4D_{B(2,3,7),c_B}[\lambda_j](\alpha_1^{d_1}), D_{Y,c_Y}[\lambda_j](\alpha_2^{d_2}) \rangle \\ &= 4D_{X,c}(\alpha_1^{d_1} \alpha_2^{d_2}) \end{aligned}$$

as required.

### 3.3. The blowup formula

In [14], formulas were given for  $D_{X\#\overline{\mathbf{CP}}^2,c}$  and  $D_{X\#\overline{\mathbf{CP}}^2,c+e}$  in terms of  $D_{X,c}$ . (Here  $e$  denotes the exceptional class.) Restricting to the case of manifolds of simple type, the following formulas were obtained.

**Theorem 3.12.** [14] *Suppose that  $X$  has simple type and  $c \in H_2(X; \mathbf{Z})$ . Then the Donaldson series of  $X\#\overline{\mathbf{CP}}^2$  are*

$$\begin{aligned} D_{X\#\overline{\mathbf{CP}}^2,c} &= D_{X,c} \exp\left(-\frac{e^2}{2}\right) \cosh(e), \\ D_{X\#\overline{\mathbf{CP}}^2,c+e} &= -D_{X,c+e} \exp\left(-\frac{e^2}{2}\right) \sinh(e). \end{aligned}$$

In particular, we note the following facts which except for (7) have been known for some time.

**Lemma 3.13.** *Let  $c \in H_2(X; \mathbf{Z})$ . Then for all  $z \in \mathbf{A}(X)$ :*

- (1)  $D_{X\#\overline{\mathbf{C}\mathbf{P}}^2,c}(e^{2k+1}z) = 0$  for all  $k \geq 0$ .
- (2)  $D_{X\#\overline{\mathbf{C}\mathbf{P}}^2,c}(z) = D_{X,c}(z)$ .
- (3)  $D_{X\#\overline{\mathbf{C}\mathbf{P}}^2,c}(e^2z) = 0$ .
- (4)  $D_{X\#\overline{\mathbf{C}\mathbf{P}}^2,c}(e^4z) = -2D_{X,c}(z)$ .
- (5)  $D_{X\#\overline{\mathbf{C}\mathbf{P}}^2,c+e}(e^{2k}z) = 0$  for all  $k \geq 0$ .
- (6)  $D_{X\#\overline{\mathbf{C}\mathbf{P}}^2,c+e}(ez) = D_{X,c}(z)$ .
- (7)  $D_{X\#\overline{\mathbf{C}\mathbf{P}}^2,c+e}(e^3z) = -D_{X,c}(zx)$ .

Item (7) was first proved by Austin and Braam [4] and Leness [26], and item (6) by Kotschick [23].

**3.4. An inductive construction of  $E(n)$**

First we establish some notation. For positive integers  $p, q$ , and  $r$ , let  $B(p, q, r)$  denote the Brieskorn manifold associated to  $p, q, r$ ; i.e., the Milnor fiber of the link of the isolated singularity of  $x^p + y^q + z^r = 0$  in  $\mathbf{C}^3$ . Then  $B(p, q, r)$  is a smooth 4-manifold with boundary, and if  $p, q$ , and  $r$  are pairwise coprime then  $\partial B(p, q, r) = \Sigma(p, q, r)$  is the corresponding Brieskorn homology 3-sphere. In general,  $b_2(B(p, q, r)) = (p-1)(q-1)(r-1)$  (cf. [5]). Two simple examples are  $B(2, 3, 5)$  which is a plumbing manifold whose intersection form is  $E_8$  (negative definite), and  $B(2, 3, 11)$  whose intersection form is  $2E_8 \oplus 2H$ .

We claim that  $E(n), n \geq 2$ , can be decomposed into three pieces

$$E(n) = B(2, 3, 11) \cup C(n) \cup B(2, 3, 6n - 11).$$

The first is the Milnor fiber  $B(2, 3, 11)$ . The second is a cobordism  $C(n)$  between  $-\Sigma(2, 3, 11)$  and  $-\Sigma(2, 3, 6n - 11)$  with intersection form

$$Q_{C(n)} = \begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix},$$

which is obtained from  $\Sigma(2, 3, 11)$  by attaching two 2-handles. In Figure 1, the result of attaching the four 2-handles with framing  $-1$  to the 4-ball has boundary  $\Sigma(2, 3, 11)$ . The cobordism  $C(n)$  is obtained by attaching the further 2-handles with framings  $0$  and  $-n$ . (In all of our handlebody pictures, the boundary of the handlebody to which we attach handles is indicated with framings that are within parentheses.) These two 2-handles represent the homology class  $f$  of the fiber and the homology class  $s_n$  of a section. The third piece is the Milnor fiber  $B(2, 3, 6n - 11)$ .

**Proposition 3.14.** *For  $n \geq 2$ ,  $B(2, 3, 11) \cup C(n) \cup B(2, 3, 6n - 11)$  is diffeomorphic to  $E(n)$ .*

*Proof.* For  $n = 2$ , this is a well-known fact [22]. The elliptic surfaces  $E(n)$ ,  $n > 2$ , can all be constructed as fiber sums,  $E(n) \cong E(2) \#_f E(n - 2)$ . Generally,  $E(m)$  has a decomposition  $E(m) = G_m \cup B(2, 3, 6m - 1)$  with the union along  $\Sigma(2, 3, 6m - 1)$  [21]. The manifold  $G_m$  with boundary  $-\Sigma(2, 3, 6m - 1)$  is known as the ‘‘Gompf nucleus’’ of  $E(m)$ . It is a regular neighborhood of the union of a cusp fiber and a section of  $E(m)$ .

Using these facts we can write our fiber sum decomposition of  $E(n)$  as

$$E(n) \cong B(2, 3, 11) \cup (G_2 \#_f G_{n-2}) \cup B(2, 3, 6n - 13).$$

There is a standard embedding of  $B(2, 3, 6n - 13)$  in  $B(2, 3, 6n - 11)$  (cf. [13]). Since  $b_2(B(2, 3, 6n - 13)) = 2(6n - 14)$  and  $b_2(B(2, 3, 6n - 11)) = 2(6n - 12)$ , the difference is a cobordism  $V(n)$  with  $b_2(V(n)) = 4$  and  $\partial V(n) = \Sigma(2, 3, 6n - 11) - \Sigma(2, 3, 6n - 13)$ . The cobordism  $V(n)$  is illustrated in Figure 2. It is composed of the two  $-1, -2$ -pairs. (The other six 2-handles when attached to the 4-ball have boundary  $\Sigma(2, 3, 6n - 11)$ .) The fiber sum  $G_2 \#_f G_{n-2}$  is shown in Figure 3 (cf. [21]). Comparing the two handlebodies we see that  $G_2 \#_f G_{n-2} = V(n) \cup C(n)$ . It follows that

$$\begin{aligned} E(n) &= B(2, 3, 11) \cup C(n) \cup V(n) \cup B(2, 3, 6n - 13) \\ &= B(2, 3, 11) \cup C(n) \cup B(2, 3, 6n - 11). \end{aligned}$$

Let  $W(n) = C(n) \cup B(2, 3, 6n - 11)$  with  $\partial W(n) = -\Sigma(2, 3, 11)$ , so that

$$E(n) = B(2, 3, 11) \cup W(n).$$

Now  $\partial B(2, 3, 11)$  bounds another interesting manifold  $C(2, 3, 11)$  obtained as the union  $C(2, 3, 11) = B(2, 3, 5) \cup D$ , where  $D$  is constructed by attaching one 2-handle to  $\Sigma(2, 3, 5)$  along the  $-1$ - framed knot in Figure 4.

**Proposition 3.15.**  $C(2, 3, 11) \cup W(n) = E(n - 1) \# \overline{\mathbf{CP}}^2$ .

*Proof.* Note that

$$E(n - 1) = B(2, 3, 5) \cup G_1 \#_f G_{n-2} \cup B(2, 3, 6n - 13),$$

so that

$$E(n - 1) \# \overline{\mathbf{CP}}^2 = B(2, 3, 5) \cup G_1 \#_f G_{n-2} \# \overline{\mathbf{CP}}^2 \cup B(2, 3, 6n - 13).$$

The cobordism  $G_1 \#_f G_{n-2} \# \overline{\mathbb{C}\mathbb{P}}^2$  is given in Figure 5 and is obtained by attaching seven 2-handles to  $\Sigma(2, 3, 5)$ . By sliding  $e$  over  $e_1$  we obtain Figure 6. Thus  $D \subset G_1 \#_f G_{n-2} \# \overline{\mathbb{C}\mathbb{P}}^2$ . The complement of  $D$  in  $G_1 \#_f G_{n-2} \# \overline{\mathbb{C}\mathbb{P}}^2$  can be seen to be  $G_2 \#_f G_{n-2}$  by sliding  $f_1$  over  $e + e_1$  and then over  $e_1$  to obtain Figure 3. Therefore

$$\begin{aligned} E(n-1) \# \overline{\mathbb{C}\mathbb{P}}^2 &= B(2, 3, 5) \cup G_1 \#_f G_{n-2} \# \overline{\mathbb{C}\mathbb{P}}^2 \cup B(2, 3, 6n-13) \\ &= B(2, 3, 5) \cup D \cup G_2 \#_f G_{n-2} \cup B(2, 3, 6n-13) \\ &= C(2, 3, 11) \cup W(n). \end{aligned}$$

In our study of  $E(3)$  and  $E(2) \# \overline{\mathbb{C}\mathbb{P}}^2$  we shall need specific information about some important homology classes in the various pieces. As usual, let  $f$  and  $S_n \in H_2(E(n); \mathbf{Z})$  denote respectively the homology classes of the fiber and a section. Let  $\bar{e} \in H^2(D; \mathbf{Z})$  denote the generator for the homology class of  $D$  so that  $\bar{e}^2 = -1$ . Figure 7 shows the cobordism from  $\Sigma(2, 3, 5)$  to  $\Sigma(2, 3, 7)$  in  $E(2) \# \overline{\mathbb{C}\mathbb{P}}^2$  arising from the embeddings  $B(2, 3, 5) \subset B(2, 3, 7) \subset E(2)$ . Sliding “ $s_2$ ” over “ $e$ ” gives Figure 8. This can also be obtained from Figure 1 with  $n = 3$  by sliding “ $\bar{e}$ ” over “ $f$ ”. This shows that  $s_2 = s_3 + e$  and that  $\bar{e} = e - f$ . Define  $\sigma_3 = 3f + 2s_3 \in H_2(E(3); \mathbf{Z})$ . It is crucial to note that  $W(3)$  is contained in both  $E(3)$  and  $E(2) \# \overline{\mathbb{C}\mathbb{P}}^2$ . Since  $f, \sigma_3 \in H_2(W(3); \mathbf{Z})$ , we may view them as elements of either  $H_2(E(3); \mathbf{Z})$  or  $H_2(E(2) \# \overline{\mathbb{C}\mathbb{P}}^2; \mathbf{Z})$ . We have  $\sigma_3^2 = 0, \sigma_3 \cdot f = 2, \sigma_3 \cdot e = 2, \sigma_3 \cdot \bar{e} = 0, e \cdot \bar{e} = -1$ , and  $f \cdot \bar{e} = 0$ .

**3.5. The low coefficients for  $E(3)$**

For most of this and the next subsection we will drop the boundary value in the notation for relative Donaldson invariants. Here, an important comment concerning the relative Donaldson invariants for  $B(2, 3, 11)$  is in order. Recall that as in Lemma 3.3, we have the generators  $\alpha, \beta$  of  $HF_*(\Sigma(2, 3, 11))$  with the grading reduced to  $\mathbf{Z}_4$ . Since  $B(2, 3, 11)$  has a big diffeomorphism group, all its homogeneous relative invariants  $D_{B(2,3,11),\omega_B}^{(d)}$  are simply multiples of powers of the intersection form. In particular, they all have even degree and boundary value  $\beta$ . Now let  $\omega_B$  be any class with  $\omega_B^2 \equiv -2 \pmod{4}$ . Then  $-1 = \hat{D}_{E(2),\omega_B}(1) = \langle D_{B(2,3,11),\omega_B}(1), D_{G_2}(1) \rangle$ ; so  $-D_{B(2,3,11),\omega_B}(1) = D_{G_2}(1) = \pm 1$ . (Recall that  $1 \in \mathbf{A}(X)$  is the unit in degree 0). We take as standard the orientation of the moduli spaces which comes from the complex structure inherited from  $E(2)$  so that  $D_{G_2}(1) = +1$ .

**Lemma 3.16.** *The relative Donaldson series of  $B(2, 3, 11)$  is*

$$D_{B(2,3,11),\omega_B} = (-1)^{\frac{1}{2}\omega_B^2} \exp(Q/2) \cdot \beta.$$

*Proof.* Let  $\alpha \in H_2(B(2, 3, 11))$ . For each  $d$ ,

$$\begin{aligned} \langle \hat{D}_{B(2,3,11),\omega_B}(\alpha^{2d}), D_{G_2}(1) \rangle &= \hat{D}_{E(2),\omega_B}(\alpha^{2d}) \\ &= (2d)!(-1)^{\frac{1}{2}\omega_B^2} \frac{Q^d}{2^d d!}(\alpha^{2d}). \end{aligned}$$

**Lemma 3.17.**  $D_{E(3)}(s_3) = 1 = -D_{E(3),s_3}(1)$ .

*Proof.* That  $D(s_3) = -D_{s_3}(1)$  follows from Theorem 3.6, and  $D_{s_3}(1) = \pm 1$  by [13]. When the moduli space  $\mathcal{M}_{s_3}$  is given the orientation induced from the Kähler structure on  $E(3)$ , the sign is '+'. Since the canonical class of  $E(3)$  is  $K_{E(3)} = f$ , this complex orientation compares with the one obtained using  $c_1 = s_3$  by  $(-1)^{\frac{1}{2}(s_3^2 + s_3 \cdot f)} = -1$ . So  $D_{s_3}(1) = -1$ .

Let  $\omega \in H_2(E(3); \mathbf{Z})$  satisfy  $\omega \cdot f \equiv 1 \pmod{2}$ . Then by Theorem 3.9,

$$(4) \quad \mathbf{D}_{E(3),\omega} = \exp(Q/2) \sum_{i=0}^{\infty} c_{2i,\omega} \frac{f^{2i}}{(2i)!}.$$

**Lemma 3.18.** *In the expression above,  $c_{0,\omega} = (-1)^{\frac{1}{2}(\omega^2 + \omega \cdot f)}$ .*

*Proof.* The group  $\text{Diff}_k(X)$  acts on the set of  $\omega \in H_2(X; \mathbf{Z}_2)$  satisfying  $\omega \cdot f \equiv 1 \pmod{2}$  with four orbits, according to whether  $\omega^2$  and  $\omega \cdot f$  are congruent to 1 or 3 (mod 4). This follows from the existence of sufficiently many spheres in  $E(3)$  of self-intersection  $-2$ ; cf. [17], [19]. If  $\sigma$  is represented by a sphere of self-intersection  $-2$  in  $B(2, 3, 11)$ , the four orbits are represented by  $\sigma + s_3, 3s_3, s_3$ , and  $\sigma + 3s_3$ . We have just seen that  $D_{E(3),s_3}(1) = -1$ . By comparing orientations,  $D_{E(3),3s_3}(1) = 1$ . Applying Theorems 3.6 and 3.5 thus yields

$$D_{E(3),\sigma+s_3}(1) = -D_{E(3),\sigma}(s_3) = -\frac{1}{2}D_{E(3)}(\sigma^2 s_3) = 2\frac{1}{2}D_{E(3)}(s_3) = 1,$$

where the next-to-last equality follows from Lemma 3.16. Again, comparing orientations shows that  $D_{E(3),\sigma+3s_3}(1) = -1$ . All these results are checked to be consistent with  $c_{0,\omega} = (-1)^{\frac{1}{2}(\omega^2 + \omega \cdot f)}$ .

Next let  $\omega \in H_2(E(3); \mathbf{Z})$  satisfy  $\omega \cdot f \equiv 0 \pmod{2}$ . Then by Theorem 3.9 we have

$$(5) \quad \mathbf{D}_{E(3),\omega} = \exp(Q/2) \sum_{i=0}^{\infty} c_{2i+1,\omega} \frac{f^{2i+1}}{(2i+1)!}.$$

**Lemma 3.19.** *Let  $\omega \in H_2(E(3); \mathbf{Z})$  with  $\omega \cdot f \equiv 0 \pmod{2}$  and  $\omega \cdot s_3 \equiv 0 \pmod{2}$ . Then  $c_{1,\omega} = (-1)^{\frac{1}{2}(\omega^2 + \omega \cdot f)}$ .*

*Proof.* By Theorem 3.6 and Lemma 3.18 we obtain

$$D_{E(3),\omega}(s_3) = -D_{E(3),\omega+s_3}(1) = -(-1)^{\frac{1}{2}(\omega+s_3) \cdot (\omega+s_3+f)} = (-1)^{\frac{1}{2}(\omega^2 + \omega \cdot f)}.$$

**3.6. The Donaldson series for  $E(n)$**

We shall now complete the proof of Theorem 3.1. Recall our notation  $\alpha, \beta$  for the generators of  $HF_*(\Sigma(2, 3, 11))$  when graded by  $\mathbf{Z}_4$ .

**Lemma 3.20.** *Suppose that  $\xi_C \in H_2(C(2, 3, 11); \mathbf{Z})$  and  $\xi_C \cdot \bar{e} \equiv 0 \pmod{2}$ . Then  $D_{C(2,3,11),\xi_C}(\bar{e}) = -(-1)^{\frac{1}{2}(\xi_C^2 + \xi_C \cdot e)} \cdot \beta$ .*

*Proof.* Write  $\xi_C = \tau + m\bar{e}$  where  $\tau \in H_2(B(2, 3, 5))$  and  $m$  is even. Since  $\bar{e} = e - f$ , we have  $\xi_C = \omega + me$  with  $\omega \in H_2(E(2); \mathbf{Z})$ . Applying Lemma 3.3 and Theorem 3.12 yields

$$\begin{aligned} \langle D_{C(2,3,11),\xi_C}(\bar{e}), D_{W(3)}(\sigma_3) \rangle &= D_{E(2)\#\overline{\mathbf{CP}}^2, \xi_C}(\bar{e}\sigma_3) \\ &= D_{E(2)\#\overline{\mathbf{CP}}^2, \omega+me}((e-f)\sigma_3) \\ &= (-1)^{\frac{m}{2}} D_{E(2),\omega}(-f(3f+2s_2)) \\ &= (-1)^{\frac{m+\omega^2}{2}}(-2) = (-1)^{\frac{1}{2}(\xi_C^2 + \xi_C \cdot e)}(-2), \end{aligned}$$

since  $m$  is even and  $\sigma_3 = 3f + 2s_2 - 2e$ .

Similarly, calculating the  $SU(2)$  invariant on  $E(3)$ , and using Lemmas 3.3, 3.19 and (5) we obtain

$$\langle D_{B(2,3,11)}(1), D_{W(3)}(\sigma_3) \rangle = D_{E(3)}(\sigma_3) = f(\sigma_3) = 2.$$

But  $D_{B(2,3,11)}(1) = \beta$ ; so we see that  $D_{W(3)}(\sigma_3) = 2\beta$ . The lemma now follows from our first equation.

**Lemma 3.21.** *Suppose that  $\xi_C \in H_2(C(2, 3, 11); \mathbf{Z})$  and  $\xi_C \cdot \bar{e} \equiv 1 \pmod{2}$ . Then we have  $D_{C(2,3,11),\xi_C}(1) = (-1)^{\frac{1}{2}(\xi_C^2 + \xi_C \cdot e)} \cdot \beta$ .*

*Proof.* As above, write  $\xi_C = \omega + m\bar{e}$  where  $\omega \in H_2(E(2); \mathbf{Z})$  and  $m$  is odd. Lemma 3.3 and Lemma 3.13 imply that

$$\begin{aligned} \langle D_{C(2,3,11),\xi_C}(1), D_{W(3)}(\sigma_3) \rangle &= D_{E(2)\#\overline{\mathbf{CP}}^2, \omega+me}(3f+2s_2-2e) \\ &= -2(-1)^{\frac{(m-1)}{2}} D_{E(2),\omega}(1) \\ &= -2(-1)^{\frac{(m-1)+\omega^2}{2}} = 2(-1)^{\frac{1}{2}(\xi_C^2 + \xi_C \cdot e)}. \end{aligned}$$

By Lemmas 3.16 and 3.19 we have

$$\langle \beta, D_{W(3)}(\sigma_3) \rangle = D_{E(3)}(\sigma_3) = 2,$$

so  $D_{W(3)}(\sigma_3) = 2 \cdot \beta$ , and the lemma follows.

Next we calculate the relative Donaldson series of  $C(2, 3, 11)$ . The relative invariant  $D_{C(2,3,11),\xi_C}$  has components  $D_{C(2,3,11),\xi_C}[\alpha]$  and  $D_{C(2,3,11),\xi_C}[\beta]$ , as well as a contribution from the trivial connection. In cases where we can apply Lemma 3.3, we can determine  $D_{C(2,3,11),\xi_C}[\beta]$ .

**Lemma 3.22.** *Let  $\xi_C \in H_2(C(2, 3, 11); \mathbf{Z})$ . If  $\xi_C \cdot \bar{e} \equiv 0 \pmod{2}$ , then*

$$\mathbf{D}_{C(2,3,11),\xi_C}[\beta] = (-1)^{\frac{1}{2}(\xi_C^2 + \xi_C \cdot e)} \exp(Q/2) \sinh(\bar{e}) \cdot \beta,$$

and if  $\xi_C \cdot \bar{e} \equiv 1 \pmod{2}$ , then

$$\mathbf{D}_{C(2,3,11),\xi_C}[\beta] = (-1)^{\frac{1}{2}(\xi_C^2 + \xi_C \cdot \bar{e})} \exp(Q/2) \cosh(\bar{e}) \cdot \beta.$$

*Proof.* First consider the case  $\xi_C \cdot \bar{e} \equiv 0 \pmod{2}$  and write  $\xi_C = \omega + me$ , with  $\omega \cdot e = 0$  and  $m$  even. Since  $\xi_C \in H_2(C(2, 3, 11))$ ,  $\xi_C \cdot f = 0$ ; so recalling that  $e = \bar{e} + f$ , we have  $\xi_C \cdot \bar{e} = \xi_C \cdot e$ . From Theorem 3.12, it follows that

$$\begin{aligned} \mathbf{D}_{E(2)\#\overline{\mathbf{CP}}^2,\xi_C} &= (-1)^{\frac{m}{2}} \mathbf{D}_{E(2),\omega} \exp(-e^2/2) \cosh(e) \\ &= (-1)^{\frac{1}{2}(m+\omega^2)} \exp(Q/2) \cosh(e) \\ &= (-1)^{\frac{1}{2}(\xi_C^2 + \xi_C \cdot \bar{e})} \exp(Q/2) \cosh(e) \\ &= (-1)^{\frac{1}{2}(\xi_C^2 + \xi_C \cdot \bar{e})} \exp(Q/2) \\ &\quad \cdot \{\cosh(\bar{e}) \cosh(f) + \sinh(\bar{e}) \sinh(f)\}. \end{aligned}$$

Note that  $\xi_C^2 \equiv \xi_C \cdot \bar{e} \equiv 0 \pmod{2}$ ; so Lemma 3.3 implies that the odd degree term in  $\bar{e}$  of this expression is

$$\langle \mathbf{D}_{C(2,3,11),\xi_C}[\beta], \mathbf{D}_{W(3)}[\beta] \rangle.$$

Thus up to a sign,  $\mathbf{D}_{C(2,3,11),\xi_C}[\beta]$  is  $\exp(Q/2) \sinh(\bar{e}) \cdot \beta$ . By Lemma 3.20, we have  $D_{C(2,3,11),\xi_C}(\bar{e}) = -(-1)^{\frac{1}{2}(\xi_C^2 + \xi_C \cdot \bar{e})} \cdot \beta$ , and therefore

$$\mathbf{D}_{C(2,3,11),\xi_C}[\beta] = (-1)^{\frac{1}{2}(\xi_C^2 + \xi_C \cdot \bar{e})} \exp(Q/2) \sinh(\bar{e}) \cdot \beta,$$

since  $\bar{e}^2 = -1$ . If  $\xi_C \cdot \bar{e} \equiv 1 \pmod{2}$ , then we similarly obtain

$$\begin{aligned} \mathbf{D}_{E(2)\#\overline{\mathbf{CP}}^2,\xi_C} &= (-1)^{\frac{1}{2}(\xi_C^2 + \xi_C \cdot e)} \exp(Q/2) \sinh(e) \\ &= (-1)^{\frac{1}{2}(\xi_C^2 + \xi_C \cdot e)} \exp(Q/2) \\ &\quad \cdot \{\cosh(\bar{e}) \sinh(f) + \sinh(\bar{e}) \cosh(f)\}, \end{aligned}$$

and from Lemma 3.3 the even degree term in  $\bar{e}$  is

$$\langle \mathbf{D}_{C(2,3,11),\xi_C}[\beta], \mathbf{D}_{W(3)}[\beta] \rangle.$$

Similarly to the other case, the result now follows from Lemma 3.21.

Next we assume inductively that Theorem 3.1 holds for  $E(n-1)$ . We now inductively determine the relative invariants of  $W(n)$ .

**Lemma 3.23.** *Let  $\eta \in H_2(W(n); \mathbf{Z})$ . If  $\eta \cdot f \equiv 0 \pmod{2}$ ,*

$$\mathbf{D}_{W(n),\eta}[\beta] = (-1)^{\frac{1}{2}(\eta^2 + (n-2)\eta \cdot f)} \exp(Q/2) \sinh^{n-2}(f) \cdot \beta,$$

*then and if  $\eta \cdot f \equiv 1 \pmod{2}$ , then*

$$\mathbf{D}_{W(n),\eta}[\beta] = (-1)^{\frac{1}{2}(\eta^2 + (n-2)\eta \cdot f)} \exp(Q/2) \cosh^{n-2}(f) \cdot \beta.$$



*Proof.* Let  $\eta \cdot f = m$ , and set  $\xi = m\bar{e} + \eta \in H_2(E(n-1) \# \overline{\mathbf{CP}}^2; \mathbf{Z})$ . Then  $\xi \cdot e = \xi \cdot (\bar{e} + f) = 0$ , and we may view  $\xi$  as an element of  $H_2(E(n-1); \mathbf{Z})$ .

Assume first that  $m \equiv 0 \pmod{2}$ . By the inductive hypothesis and Theorem 3.12,

$$\begin{aligned} \mathbf{D}_{E(n-1) \# \overline{\mathbf{CP}}^2, \xi} &= (-1)^{\frac{1}{2}(\xi^2 + (n-3)\xi \cdot f)} \exp(Q/2) \sinh^{n-3}(f) \cosh(e) \\ &= (-1)^{\frac{1}{2}(\xi^2 + (n-3)\xi \cdot f)} \exp(Q/2) \{ \sinh^{n-3}(f) \cosh(f) \cosh(\bar{e}) \\ &\quad + \sinh^{n-2}(f) \sinh(\bar{e}) \}. \end{aligned}$$

Applying Lemma 3.3, we see that the odd degree term in  $\bar{e}$  of the above expression is

$$\langle \mathbf{D}_{C(2,3,11), m\bar{e}}[\beta], \mathbf{D}_{W(3), \eta}[\beta] \rangle.$$

By Lemma 3.22,

$$\begin{aligned} \mathbf{D}_{C(2,3,11), m\bar{e}}[\beta] &= (-1)^{\frac{m}{2}} \exp(Q/2) \sinh(\bar{e}) \cdot \beta \\ &= (-1)^{\frac{1}{2}(\eta \cdot f)} \exp(Q/2) \sinh(\bar{e}) \cdot \beta. \end{aligned}$$

Hence  $\mathbf{D}_{W(3), \eta}[\beta] = (-1)^{\frac{1}{2}(\eta^2 + (n-2)\eta \cdot f)} \exp(Q/2) \sinh^{n-2}(f) \cdot \beta$ .

Next, if  $\eta \cdot f = m \equiv 1 \pmod{2}$ , then

$$\begin{aligned} \mathbf{D}_{E(n-1) \# \overline{\mathbf{CP}}^2, \xi} &= (-1)^{\frac{1}{2}(\xi^2 + (n-3)\xi \cdot f)} \exp(Q/2) \cosh^{n-3}(f) \cosh(e) \\ &= (-1)^{\frac{1}{2}(\xi^2 + (n-3)\xi \cdot f)} \exp(Q/2) \\ &\quad \cdot \{ \cosh^{n-2}(f) \cosh(\bar{e}) + \cosh^{n-3}(f) \sinh(f) \sinh(\bar{e}) \}. \end{aligned}$$

This time, Lemma 3.3 implies that the even degree term in  $\bar{e}$  in the above expression is

$$\langle \mathbf{D}_{C(2,3,11), m\bar{e}}[\beta], \mathbf{D}_{W(3), \eta}[\beta] \rangle,$$

and the lemma follows as in the other case.

By Lemma we have for any  $\omega_B \in H_2(B(2, 3, 11); \mathbf{Z})$ ,

$$\mathbf{D}_{B(2,3,11), \omega_B} = (-1)^{\frac{1}{2}\omega_B^2} \exp(Q/2) \cdot \beta.$$

Thus for any  $\omega = \omega_B + \eta \in H_2(E(n); \mathbf{Z})$ ,

$$\mathbf{D}_{E(n), \omega} = \langle (-1)^{\frac{1}{2}\omega_B^2} \exp(Q/2) \cdot \beta, \mathbf{D}_{W(n), \eta}[\beta] \rangle.$$

If  $\omega \cdot f \equiv 0 \pmod{2}$ , then

$$\begin{aligned} \mathbf{D}_{E(n),\omega} &= \langle (-1)^{\frac{1}{2}\omega_B^2} \exp(Q/2) \cdot \beta, (-1)^{\frac{1}{2}(\eta^2+(n-2)\eta \cdot f)} \\ &\quad \cdot \exp(Q/2) \sinh^{n-2}(f) \cdot \beta \rangle \\ &= (-1)^{\frac{1}{2}(\omega^2+(n-2)\omega \cdot f)} \exp(Q/2) \sinh^{n-2}(f), \end{aligned}$$

and if  $\omega \cdot f \equiv 1 \pmod{2}$ , then

$$\begin{aligned} \mathbf{D}_{E(n),\omega} &= \langle (-1)^{\frac{1}{2}\omega_B^2} \exp(Q/2) \cdot \beta, (-1)^{\frac{1}{2}(\eta^2+(n-2)\eta \cdot f)} \\ &\quad \cdot \exp(Q/2) \cosh^{n-2}(f) \cdot \beta \rangle \\ &= (-1)^{\frac{1}{2}(\omega^2+(n-2)\omega \cdot f)} \exp(Q/2) \cosh^{n-2}(f). \end{aligned}$$

This completes the proof of Theorem 3.1.

#### 4. Relations for embedded 2-spheres

The goal of this section is to prove the following theorem, which is the foundation of the theory expounded in this paper.

**Theorem 4.1.** *Let  $X$  be an oriented simply connected 4-manifold of simple type, which contains an embedded 2-sphere  $S$  representing an homology class  $\sigma$  with self-intersection  $\sigma^2 \leq -2$ . Then there are constants  $A_{i,k}$  and  $B_{j,k}$  depending only on  $\sigma^2$ , such that for  $z \in \mathbf{A}(\sigma^\perp)$ ,*

$$\begin{aligned} \hat{D}(\sigma^{2k} z) &= \sum_{j=1}^k A_{j,k} \hat{D}(\sigma^{2k-2j} z) + A_{k+1,k} \hat{D}_\sigma(z) \quad \text{if } \sigma^2 = -(2k), \text{ and} \\ \hat{D}(\sigma^{2k-1} z) &= \sum_{i=1}^{k-1} B_{i,k} \hat{D}(\sigma^{2k-2i-1} z) + B_{k,k} \hat{D}_\sigma(z) \quad \text{if } \sigma^2 = -(2k+1). \end{aligned}$$

As we have mentioned earlier, the case  $\sigma^2 = -2$  is due to D. Ruberman [32]. His result served, in part, to motivate the above theorem.

Consider first the case where  $\sigma^2 = -(2k+1)$ . Let  $N$  be a tubular neighborhood of  $S$ . It is the 2-disk bundle over  $S^2$  of degree  $-(2k+1)$ , and  $\partial N$  is the lens space  $L = L(2k+1, -1)$ . Let  $X_0$  be the closure of  $X \setminus N$ ; so  $X = N \cup X_0$  and  $\partial X_0 = \bar{L} = L(2k+1, 1)$ . Let  $\zeta$  be the generator of the character variety  $\chi_{SU(2)}(L)$  of  $SU(2)$  representations of  $\pi_1(L)$  mod conjugacy. For  $1 \leq m \leq k$  the  $\rho$ -invariant of  $\zeta^m$  is

$$\rho(\zeta^m) = 2 - 8m + \frac{16m^2}{2k+1},$$

(see e.g. [28]) and  $\rho(\zeta^0) = 0$ .

It will be important to understand the anti-self-dual reducible connections on  $N$  (with an infinite cylindrical end  $L \times [0, \infty)$ ). Let  $\lambda$  be the complex line bundle over  $N$  with  $\langle c_1(\lambda), \sigma \rangle = -1$ , and let  $\mathcal{M}_N(\lambda^\ell \oplus \bar{\lambda}^\ell)$  be the moduli space of finite action anti-self-dual connections on  $\lambda^\ell \oplus \bar{\lambda}^\ell$ . Such connections are asymptotically flat, and their boundary value is conjugate to  $\zeta^\ell$ . Let  $\delta(\ell)$  be the dimension of the stabilizer of  $\zeta^\ell$ . Since  $\sigma^2 = -(2k + 1)$ , we have for  $1 \leq \ell \leq k$ ,  $\delta(\ell) = 1$ , and  $\delta(0) = 3$ . Furthermore, the Atiyah, Patodi, Singer's Theorem [2] gives

$$\dim \mathcal{M}_N(\lambda^\ell \oplus \bar{\lambda}^\ell) = 8c_2(\lambda^\ell \oplus \bar{\lambda}^\ell) - \frac{3}{2}(e(N) + \text{sign}(N)) - \frac{\delta(\ell) + \rho(\zeta^\ell)}{2} = 4\ell - 3$$

for  $0 \leq \ell \leq k$ . Here,  $c_2(\lambda^\ell \oplus \bar{\lambda}^\ell) = \frac{\ell^2}{2k+1}$ .

For a fixed  $\kappa \leq k$ , we wish to calculate  $D(\sigma^{2\kappa-1}z)$  for an arbitrary monomial  $z$  sitting, say, in degree  $d$  in the graded algebra  $\mathbf{A}(\sigma^\pm)$  (and so that, say,  $\deg D \equiv 2\kappa - 1 + \deg z \pmod{4}$ ). To make this calculation we shall use important techniques due to Cliff Taubes [35], [36], [37], [38]. As we have stated in the introduction, an alternative approach is provided by the thesis of W. Wieczorek [39]. His technique involves a partition of the compactified moduli space into compact domains equipped with fixed framings of the basepoint fibration over the boundaries. Using this partition, he is able to prove Propositions 4.5, 4.6 below by evaluating relative cohomology classes.

First we need the following notation:

$\mathcal{M}_{N,b}[i]$  denotes the cylindrical end moduli space consisting of finite action anti-self-dual connections on  $N$  asymptotic to  $\zeta^i$  and of dimension  $4i - 3 + 8b$ ,  $b \geq 0$ .

$\mathcal{M}_{X_0,\kappa,b}[j]$  denotes the cylindrical end moduli space consisting of finite action anti-self-dual connections on  $X_0$  asymptotic to  $\zeta^j$  and of complementary dimension

$$(2d + 4\kappa - 2) - (4j - 3 + 8b + \delta(j)), \quad b \geq 0.$$

$\mathcal{M}_{L,t}[i, j]$  denotes the cylindrical end moduli space consisting of finite action anti-self-dual connections on  $L \times \mathbf{R}$  asymptotic to  $\zeta^i$  at  $-\infty$  and to  $\zeta^j$  at  $+\infty$  and (by Lemma 4.2 below) of dimension  $4(j - i) - \delta(i) + 8t$ ,  $t \geq 0$ .

Since  $\mathcal{M}_{N,0}[i] = \mathcal{M}_N(\lambda^i \oplus \bar{\lambda}^i)$  and instantons can be grafted into this moduli space, the moduli space  $\mathcal{M}_{N,b}[i]$  is nonempty and contains a reducible anti-self-dual connection. All connections in these moduli spaces decay exponentially at  $\infty$  (and at  $-\infty$ ). The moduli spaces

$\mathcal{M}_{L,t}[i, j]$  were studied in David Austin's thesis [3] and by Furuta and Hashimoto [20].

**Lemma 4.2.** *For  $0 \leq i, j \leq k$ , any nonempty moduli space on  $L \times \mathbf{R}$  with boundary values  $\zeta^i$  at  $-\infty$  and  $\zeta^j$  at  $+\infty$  has dimension  $4(j - i) - \delta(i) + 8t$  for some  $t \geq 0$ .*

*Proof.* Let  $\mathcal{M}_L[i, j]$  be the moduli space in question, and first assume that the reduced moduli space  $\mathcal{M}'_L[i, j] = \mathcal{M}_L[i, j]/\mathbf{R}$ , the quotient by translations, is compact. According to [3], [20] each compact  $\mathcal{M}'_L[i, j]$  is either 0 or 2-dimensional. From [3, §§4.2, 6.2] we see that if the compact moduli space  $\mathcal{M}'_L[i, j]$  is nonempty, then it comes from a bundle over  $L \times \mathbf{R}$  which has  $c_2 = -ab/(2k + 1)$  where  $a = j + i$  and  $b = j - i$ . If we write  $j = i + r$ , then  $c_2 = -r(2i + r)/(2k + 1)$ . Using the  $\rho$ -invariants of  $L$  given above, the index theorem [2] gives  $\dim \mathcal{M}'_L[i, i + r] = 4r - \delta(i) - 1$ . When  $i \neq 0$ , this moduli space can be compact only when  $r = 1$  (and it is 2-dimensional). When  $i = 0$ , if compact, then  $\dim \mathcal{M}'_L[0, j] = 4j - 4$ ; so  $j = 1$ , and  $\mathcal{M}'_L[0, 1]$  is 0-dimensional.

To prove the theorem in general, use the fact (see [16]) that for any  $\mathcal{M}_L[i, j]$  we can expand an end of  $\mathcal{M}'_L[i, j]$  into compact pieces; i.e., write an end locally as

$$\mathcal{M}'_L[\ell_0, \ell_1] \times G(\ell_1) \times \mathbf{R} \times \dots \times G(\ell_{t-1}) \times \mathbf{R} \times \mathcal{M}'_L[\ell_{t-1}, \ell_t],$$

where  $\ell_0 = i$ ,  $\ell_t = j$ , each  $\mathcal{M}'_L[\ell_{i-1}, \ell_i]$  is compact, and  $G(\ell_i) = \text{Stab}(\zeta^{\ell_i})$ . If  $0 \leq i < j \leq k$ , then the shortest such path is given by  $\ell_m = i + m$ ,  $t = j - i$ , and the theorem follows.

This lemma does not depend on the fact that  $\pi_1(L)$  has odd order.

To apply Taubes' techniques, we study the cylindrical end based moduli spaces  $\tilde{\mathcal{M}}_{N,b}[i]$  and  $\tilde{\mathcal{M}}_{X_0,\kappa,b}[i]$  of anti-self-dual connections modulo gauge transformations which are asymptotic to the identity. Here, the basepoint is taken at  $\infty$ ; this is the same as considering the orbifold obtained by coning off the lens space  $L$  and using the cone point as basepoint. If  $\delta(i) = 1$ , we have the boundary value map

$$\partial_{N,b}[i] : \tilde{\mathcal{M}}_{N,b}[i] \rightarrow S^2[i],$$

where  $S^2[i] \subset SO(3)$  is the conjugacy class  $\zeta^i$  of representations of  $\pi_1(L)$  on  $SO(3) = SU(2)/\text{center}$ , and similarly we have  $\partial_{X_0,\kappa,b}[i]$ . If  $\delta(i) = 3$  then the map  $\partial_{N,b}[i]$  is the (trivial) map to  $\{1\} \subset SO(3)$ . Recall that to define  $\partial_{N,b}[i]$  at some  $A \in \tilde{\mathcal{M}}_{N,b}[i]$ , one needs to trivialize  $P$  over the cylinder, and then look at the limit of the restriction of  $A$  over  $L \times \{t\}$  as  $t \rightarrow \infty$  ([28]).

Using the boundary value maps we form fiber products  $\tilde{\mathcal{M}}_{N,b}[i] \times_i \tilde{\mathcal{M}}_{X_0,\kappa,b}[i]$  over  $S^2[i]$  or  $\{1\}$ , as the case may be. The boundary value

maps satisfy the following compatibility properties. (See e.g. [35], [36].)

**Proposition 4.3.** *The boundary value maps  $\partial_{N,b}[i]$ ,  $\partial_{L,t}[i]$ ,  $\partial_{L,t}[j]$ , and  $\partial_{X_0,\kappa,b}[j]$  are continuous maps. Furthermore,  $\tilde{\mathcal{M}}_{N,b}[i] \times_i \tilde{\mathcal{M}}'_{L,t}[i, j]$  can be identified with a subset of  $\tilde{\mathcal{M}}_{N,b+t}[j]$ , and  $\tilde{\mathcal{M}}'_{L,t_1}[i, j] \times_j \tilde{\mathcal{M}}'_{L,t_2}[j, \ell]$  can be identified with a subset of  $\tilde{\mathcal{M}}'_{L,t_1+t_2}[i, \ell]$ , etc. The boundary value maps are compatible with these identifications in that the restriction of  $\partial_{N,b+t}[j]$  to the image of  $\tilde{\mathcal{M}}_{N,b}[i] \times_i \tilde{\mathcal{M}}'_{L,t}[i, j]$  is equal to the map induced from  $\partial_{L,t}[j]$ , etc. Furthermore, there are identifications*

$$\begin{aligned} &(\tilde{\mathcal{M}}'_{L,t_1}[i, j] \times_j \tilde{\mathcal{M}}'_{L,t_2}[j, \ell]) \times_\ell \tilde{\mathcal{M}}'_{L,t_3}[\ell, m] \\ &\cong \tilde{\mathcal{M}}'_{L,t_1}[i, j] \times_j (\tilde{\mathcal{M}}'_{L,t_2}[j, \ell] \times_\ell \tilde{\mathcal{M}}'_{L,t_3}[\ell, m]), \end{aligned}$$

as subsets of  $\tilde{\mathcal{M}}'_{L,t_1+t_2+t_3}[i, m]$ , compatible with the boundary value maps.

From the gluing theorems of Mrowka [31], Morgan and Mrowka [30], and Taubes [34], it follows that

$$\tilde{\mathcal{M}}_X \cong \bigcup_i \bigcup_b \tilde{\mathcal{M}}_{N,b}[i] \times_i \tilde{\mathcal{M}}_{X_0,\kappa,b}[i].$$

Thus, letting  $\tau$  be a 3-form which integrates to 1 over the fibers of the basepoint fibration over  $\mathcal{M}_{X_0}$ , one needs to integrate

$$(6) \quad \int_{\tilde{\mathcal{M}}_{N,b}[i] \times_i \tilde{\mathcal{M}}_{X_0,\kappa,b}[i]} \tau \wedge \tilde{\mu}(\sigma)^{2\kappa-1} \wedge \tilde{\mu}(z).$$

The form  $\tilde{\mu}(z)$  on  $\tilde{\mathcal{M}}_{X_0,\kappa,b}[i]$  pulls back from the form  $\mu(z)$  on  $\mathcal{M}_{X_0,\kappa,b}[i]$  which contains no reducible connections. However, since  $\delta(i) = \dim(\text{Stab})(\zeta^i) \geq 1$ , it follows that  $\dim \mathcal{M}_{N,b}[i] < 4\kappa - 2$ ; so  $\tilde{\mu}(\sigma)$  does not pull back from  $\mathcal{M}_{N,b}[i]$ . (Otherwise we would have  $\tilde{\mu}(\sigma)^{2\kappa-1} = 0$  by dint of a dimension count.) This means that the  $SO(3)$  action on  $\tilde{\mathcal{M}}_{N,b}[i]$  which gives the basepoint map is not free. (In order to make this argument precise, first one needs to blow up  $X$  at a point in  $X_0$  and then to use the relation  $D_{X \# \overline{\mathbb{C}P}^2, e}(\sigma^{2\kappa-1}ze) = D_X(\sigma^{2\kappa-1}z)$  [29] as we mentioned in §2. Since there are no flat connections on  $X_0 \# \overline{\mathbb{C}P}^2$  with  $w_2 \equiv e \pmod{2}$ , the counting argument goes through unencumbered. We assume this done, and for simplicity we do not change notation.)

Now  $\tilde{\mu}(\sigma)^{2\kappa-1}$  vanishes near the trivial connection; so if our integral is to be nonzero,  $\tilde{\mathcal{M}}_{N,b}[i]$  must contain a sequence of connections which converges weakly to a nontrivial reducible connection. This situation has also been studied by Taubes in [36]. Although  $\tilde{\mu}(\sigma)$  does not pull back from the base, it does define an  $SO(3)$  equivariant cohomology class

as follows. If  $m : SO(3) \times \tilde{\mathcal{M}}_{N,b}[i] \rightarrow \tilde{\mathcal{M}}_{N,b}[i]$  is the  $SO(3)$  action, and  $p : SO(3) \times \tilde{\mathcal{M}}_{N,b}[i] \rightarrow \tilde{\mathcal{M}}_{N,b}[i]$  is projection on the second factor, then there is a smooth form  $\omega$  on  $SO(3) \times \tilde{\mathcal{M}}_{N,b}[i]$  such that  $m^*(\tilde{\mu}(\sigma)) = p^*(\tilde{\mu}(\sigma)) + d\omega$ . This allows the construction of an  $SO(3)$ -equivariant extension  $\hat{\mu}(\sigma)$ . As explained in [36] (also cf. [4]), it is important to note that the construction of the extended form  $\hat{\mu}(\sigma)$  cannot be made solely in terms of data derived from  $N$ . From  $X_0$  one needs the choice of a connection on the principal  $SO(3)$  bundle  $\tilde{\mathcal{M}}_{X_0,\kappa,b}[i] \rightarrow \mathcal{M}_{X_0,\kappa,b}[i]$ . Taubes has shown [37] that there is an enlargement of the moduli space  $\tilde{\mathcal{M}}_{N,b}[i] \times_i \tilde{\mathcal{M}}_{X_0,\kappa,b}[i]$  which has a compactification as a manifold with corners, and the forms  $\tilde{\mu}(\sigma)$ ,  $\tilde{\mu}(z)$ , and  $\tau$  are push-forwards of forms which extend over this compactification. Furthermore, the pullback of the top-dimensional form  $\tau \wedge \tilde{\mu}(\sigma)^{2\kappa-1} \wedge \tilde{\mu}(z)$  pulls back from lower strata when restricted over the corners. Thus the integral (6) is finite and well-defined.

According to [37], [38], the push-forward map  $(\partial_{N,b}[i])_*$  to equivariant cohomology is well-defined, and

$$\begin{aligned} & \int_{\tilde{\mathcal{M}}_{N,b}[i] \times_i \tilde{\mathcal{M}}_{X_0,\kappa,b}[i]} \tau \wedge \tilde{\mu}(z) \wedge \tilde{\mu}(\sigma)^{2\kappa-1} \\ &= \int_{\tilde{\mathcal{M}}_{X_0,\kappa,b}[i]} \tau \wedge \tilde{\mu}(z) \wedge (\partial_{X_0,\kappa,b}[i])^*(\partial_{N,b}[i])_*(\tilde{\mu}(\sigma)^{2\kappa-1}), \end{aligned}$$

where  $(\partial_{X_0,\kappa,b}[i])^*$  denotes pullback from equivariant cohomology. Taubes has shown by similar considerations as above that this last integral is well-defined.

For  $i = 0$ ,  $\partial_{N,b}[0] : \tilde{\mathcal{M}}_{N,b}[0] \rightarrow \{1\}$  has fiber dimension equal to  $\dim \tilde{\mathcal{M}}_{N,b}[0] = 8b$ . So the cohomology class represented by the cocycle  $(\partial_{N,b}[i])_*(\tilde{\mu}(\sigma)^{2\kappa-1})$  is 0, because it lives in

$$H_{SO(3)}^{4\kappa-8b-2}(\{1\}; \mathbf{R}) = H^{4\kappa-8b-2}(BSO(3); \mathbf{R}) = 0.$$

If  $1 \leq i \leq k$ , then  $\partial_{N,b}[i] : \tilde{\mathcal{M}}_{N,b}[i] \rightarrow S^2[i]$  has fiber dimension  $4i + 8b - 2$ , and

$$(\partial_{N,b}[i])_*(\hat{\mu}(\sigma)^{2\kappa-1}) \in H_{SO(3)}^{4(\kappa-i)-8b}(S^2[i]; \mathbf{R}).$$

But  $H_{SO(3)}^*(S^2[i]) = H^*(\mathbf{CP}^\infty)$  is generated by a 2-dimensional cocycle  $v$ . This means that  $(\partial_{N,b}[i])_*(\hat{\mu}(\sigma)^{2\kappa-1})$  is some multiple  $m_k(\kappa, i, b)v^{2(\kappa-i)-4b}$ . Notice that this implies

$$0 < i \leq \kappa \quad \text{and} \quad b \leq \frac{\kappa - i}{2}.$$

To calculate  $(\partial_{X_0,\kappa,b}[i])^*(v)$ , note that the fact that we have  $\partial_{X_0,\kappa,b}[i] : \tilde{\mathcal{M}}_{X_0,\kappa,b}[i] \rightarrow S^2[i]$  implies that the basepoint fibration  $\beta_{X_0,\kappa,b}[i] :$

$\tilde{\mathcal{M}}_{X_0, \kappa, b}[i] \rightarrow \mathcal{M}_{X_0, \kappa, b}[i]$  reduces to a principal  $S^1$  bundle

$$\gamma_{X_0, \kappa, b}[i] : (\partial_{X_0, \kappa, b}[i])^{-1}(s_i) \rightarrow \mathcal{M}_{X_0, \kappa, b}[i],$$

for a point  $s_i \in S^2[i]$ . Now  $(\partial_{X_0, \kappa, b}[i])^*(v)$  represents a class in  $H^2_{SO(3)}(\tilde{\mathcal{M}}_{X_0, \kappa, b}[i]; \mathbf{R}) \cong H^2(\mathcal{M}_{X_0, \kappa, b}[i]; \mathbf{R})$  since the  $SO(3)$ -action on  $\tilde{\mathcal{M}}_{X_0, \kappa, b}[i]$  is free. The following fact is pointed out by Austin and Braam in [4] where a differential-geometric proof is indicated.

**Lemma 4.4.** *Under the identification of  $H^2_{SO(3)}(\tilde{\mathcal{M}}_{X_0, \kappa, b}[i]; \mathbf{R})$  with  $H^2(\mathcal{M}_{X_0, \kappa, b}[i]; \mathbf{R})$ , the pullback  $(\partial_{X_0, \kappa, b}[i])^*(v)$  represents  $\varepsilon$ , the Euler class of the  $S^1$  fibration  $\gamma_{X_0, \kappa, b}[i]$ .*

*Proof.* One can factor the classifying map of  $\gamma_{X_0, \kappa, b}[i]$  as follows:

$$\begin{array}{ccccccc} \partial^{-1}(s_i) & \rightarrow & \partial^{-1}(s_i) \times ESO(2) & \rightarrow & \{s_i\} \times ESO(2) & \xrightarrow{\sim} & ESO(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_{X_0, \kappa, b}[i] & \xrightarrow{\sim} & \partial^{-1}(s_i) \times_{SO(2)} ESO(2) & \xrightarrow{\partial} & \{s_i\} \times_{SO(2)} ESO(2) & \xrightarrow{\sim} & BSO(2) \end{array}$$

This pulls back the universal class  $v$  to  $\partial^*v$ .

We now have

$$\begin{aligned} & \int_{\tilde{\mathcal{M}}_{N, b}[i] \times_i \tilde{\mathcal{M}}_{X_0, \kappa, b}[i]} \tau \wedge \tilde{\mu}(z) \wedge \tilde{\mu}(\sigma)^{2\kappa-1} \\ &= m_k(\kappa, i, b) \int_{\tilde{\mathcal{M}}_{X_0, \kappa, b}[i]} \tau \wedge \tilde{\mu}(z) \wedge (\partial_{X_0, \kappa, b}[i])^*(v^{2(\kappa-i)-4b}) \\ &= m_k(\kappa, i, b) \int_{\mathcal{M}_{X_0, \kappa, b}[i]} \mu(z) \wedge \varepsilon^{2(\kappa-i)-4b} \\ &= M_k(\kappa, i, b) \int_{\mathcal{M}_{X_0, \kappa, b}[i]} \mu(z) \wedge \nu^{\kappa-i-2b}, \end{aligned}$$

where  $M_k(\kappa, i, b) = (-4)^{\kappa-i-2b} m_k(\kappa, i, b)$ , since  $\varepsilon^2 = p_1(\beta_{X_0, \kappa, b}[i])$  and  $\nu = -\frac{1}{4}p_1(\beta_{X_0, \kappa, b}[i])$ . The integral

$$\int_{\mathcal{M}_{X_0, \kappa, b}[i]} \mu(z) \wedge \nu^{\kappa-i-2b} = D_{X_0}[i](z x^{\kappa-i-2b})$$

is a relative invariant on  $X_0$ . Alternatively, one may view this invariant as a Donaldson invariant of the orbifold obtained by collapsing the 2-sphere  $S \subset X$  to a point and using bundles with rotation number  $i$  over the cone point.

So far we have contributions

$$\sum_{i=1}^{\kappa} \sum_{b=0}^{\lfloor \frac{\kappa-i}{2} \rfloor} M_k(\kappa, i, b) D_{X_0}[i](z x^{\kappa-i-2b})$$

to the calculation of  $D(\sigma^{2\kappa-1}z)$ . This does not give the complete calculation because multiple counting may have occurred. For example, the double fiber product

$$\tilde{\mathcal{M}}_{N,b}[i] \times_i \tilde{\mathcal{M}}'_{L,t}[i,j] \times_j \tilde{\mathcal{M}}_{X_0,\kappa,b+t}[j]$$

is contained in both  $\tilde{\mathcal{M}}_{N,b}[i] \times_i \tilde{\mathcal{M}}_{X_0,\kappa,b}[i]$  and  $\tilde{\mathcal{M}}_{N,b+t}[j] \times_j \tilde{\mathcal{M}}_{X_0,\kappa,b+t}[j]$ . Thus we need to appropriately add or subtract contributions from higher order fiber products. For example, computing as above,

$$\begin{aligned} & \int_{\tilde{\mathcal{M}}_{N,b}[i] \times_i \tilde{\mathcal{M}}'_{L,t}[i,j] \times_j \tilde{\mathcal{M}}_{X_0,\kappa,b+t}[j]} \tau \wedge \tilde{\mu}(z) \wedge \tilde{\mu}(\sigma)^{2\kappa-1} \\ &= m \int_{\tilde{\mathcal{M}}'_{L,t}[i,j] \times_j \tilde{\mathcal{M}}_{X_0,\kappa,b+t}[j]} \tau \wedge \tilde{\mu}(z) \wedge (\partial_{L,t}[i])^*(v^{2(\kappa-i)-4b}) \\ &= mn \int_{\tilde{\mathcal{M}}_{X_0,\kappa,b+t}[j]} \tau \wedge \tilde{\mu}(z) \wedge (\partial_{X_0,\kappa,b+t}[j])^*(v^{2(\kappa-j)-4(b+t)}), \end{aligned}$$

since the fiber dimension of  $\partial_{L,t}[j]$  is  $4(j-i)+8t$ , and where the constants  $m, n$  depend only on  $\sigma^2, \kappa, i, j, b$ , and  $t$ . This is then a multiple of  $D_{X_0}[j](z x^{\kappa-j-2(b+t)})$  as before. Adding (or subtracting as necessary) all these correction terms, we obtain

**Proposition 4.5.** *Let  $\sigma^2 = -(2k + 1)$  and  $0 < \kappa \leq k$ . Then for all  $z \in \mathbf{A}(\sigma^\perp)$ ,*

$$D_X(\sigma^{2\kappa-1}z) = \sum_{i=1}^{\kappa} \sum_{b=0}^{\lfloor \frac{\kappa-i}{2} \rfloor} r_k(\kappa, i, b) D_{X_0}[i](z x^{\kappa-i-2b}),$$

where the coefficients  $r_k(\kappa, i, b)$  depend only on  $\sigma^2, \kappa, i$ , and  $b$ .

In fact, one might reasonably attribute this last proposition to Taubes [37], [38]; we have simply provided a reader's guide. (Cf. also [39].)

In the even case,  $\sigma^2 = -2k$ , we need to evaluate  $D(\sigma^{2k}z)$ . Our above arguments work in this case with very minor modifications. First, the argument which rules out the trivial boundary value no longer applies. In this case, the boundary value map  $\partial_{N,b}[0] : \tilde{\mathcal{M}}_{N,b}[0] \rightarrow \{1\}$  has fiber dimension  $8b$ ; so  $(\partial_{N,b}[0])_* (\tilde{\mu}(\sigma)^{2\kappa}) \in H_{SO(3)}^{4\kappa-8b}(\{1\}; \mathbf{R}) = \mathbf{R}$ . Since  $H_{SO(3)}^*(\{1\}; \mathbf{R}) = H^*(BSO(3); \mathbf{R})$  is a polynomial algebra generated by a 4-dimensional cocycle  $u$ , we get  $(\partial_{N,b}[0])_* (\tilde{\mu}(\sigma)^{2\kappa}) = m'_k(\kappa, 0, b) u^{\kappa-2b}$ . By a dimension counting argument, the trivial moduli space of formal dimension  $-3$  does not contribute to the calculation of  $D(z\sigma^{2k})$  if  $\kappa > 0$ . Furthermore,  $\partial_{X_0,\kappa,b}[0]^*(u) = p_1(\beta_{X_0})$ . So one gets terms  $D_{X_0}[0](z x^{\kappa-2b})$  in the calculation for  $b \geq 1$  and  $\kappa \geq 2$ . For  $0 < i < \kappa$ , the fiber dimension of  $\partial_{N,b}[i] : \tilde{\mathcal{M}}_{N,b}[i] \rightarrow S^2[i]$  is  $4i + 8b - 2$ ; so



$(\partial_{N,b}[i])_*(\tilde{\mu}(\sigma)^{2\kappa}) = m'_k(\kappa, i, b)v^{2(\kappa-i)-4b+1}$ . This means that we have to integrate terms of the form

$$\begin{aligned} & \int_{\tilde{\mathcal{M}}_{X_0, \kappa, b}[i]} \tau \wedge \tilde{\mu}(z) \wedge (\partial_{X_0, \kappa, b}[i])^*(v^{2(\kappa-i)-4b+1}) \\ &= \int_{\mathcal{M}_{X_0, \kappa, b}[i]} \mu(z) \wedge \varepsilon^{2(\kappa-i)-4b+1} \\ &= (-4)^{\kappa-i-2b} \int_{\mathcal{M}_{X_0, \kappa, b}[i]} \mu(z) \wedge \nu^{\kappa-i-2b} \wedge \varepsilon, \end{aligned}$$

since  $\varepsilon^2 = p_1(\beta) = -4\nu$ . For obvious reasons, we shall denote this last integral as  $D_{X_0}[i](z x^{\kappa-i-2b+\frac{1}{2}})$ . For  $i = k$ , we have  $\delta(k) = 3$ , and arguing similarly to the case  $i = 0$  we see that the only contribution to the calculation is  $m'_k(k, i, b)D_{X_0}[k](z)$ .

**Proposition 4.6.** *Let  $\sigma^2 = -2k$  and  $0 < \kappa \leq k$ . For  $z \in \mathbf{A}(\sigma^\perp)$ , if  $\kappa < k$ , then*

$$\begin{aligned} D_X(\sigma^{2\kappa} z) &= \sum_{i=1}^{\kappa} \sum_{b=0}^{\lfloor \frac{\kappa-i}{2} \rfloor} s_k(\kappa, i, b) D_{X_0}[i](z x^{\kappa-i-2b+\frac{1}{2}}) \\ &\quad + \sum_{b=0}^{\lfloor \frac{\kappa}{2} \rfloor} s_k(\kappa, 0, b) D_{X_0}[0](z x^{\kappa-2b}), \end{aligned}$$

and

$$\begin{aligned} D_X(\sigma^{2k} z) &= s_k(k, k, 0) D_{X_0}[k](z) + \sum_{i=1}^{k-1} \sum_{b=0}^{\lfloor \frac{k-i}{2} \rfloor} s_k(k, i, b) D_{X_0}[i](z x^{k-i-2b+\frac{1}{2}}) \\ &\quad + \sum_{b=0}^{\lfloor \frac{k}{2} \rfloor} s_k(k, 0, b) D_{X_0}[0](z x^{k-2b}) \end{aligned}$$

where the coefficients  $s_k(\kappa, i, b)$  depend only on  $\sigma^2$ ,  $\kappa$ ,  $i$ , and  $b$ , and if  $\kappa > 0$  then  $s_k(\kappa, 0, 0) = 0$ .

We are now in a position to prove Theorem 4.1. Consider first  $\sigma^2 = -(2k + 1)$ . From Proposition 4.5 we have  $D(\sigma z) = r_k(1, 1, 0)D_{X_0}[1](z)$ . We claim that  $r_k(1, 1, 0) \neq 0$ . To see this let  $X$  be  $E(3) \# (2k - 2)\overline{\mathbf{CP}}^2$  with  $f$  a fiber class and  $s$  a section class in  $E(3)$ , and let  $e_i$ ,  $i = 1, \dots, 2k - 2$ , be the exceptional classes. Then  $\sigma = s - \sum_{i=1}^{2k-2} e_i$  is represented by an embedded 2-sphere in  $X$ , and  $\sigma^2 = -(2k + 1)$ . From Theorems 3.1 and 3.12 it follows that the  $SU(2)$  Donaldson series of  $X$  is  $\mathbf{D}_X = \exp(Q/2) \sinh(f) \prod \cosh(e_i)$ . In general,  $\frac{1}{2}(1 + b_{E(n)}^+) \equiv n \pmod{4}$ ; so  $D_X$  can be nonzero only on elements of  $\mathbf{A}(X)$  in degrees

congruent to 3 (mod 4). Thus  $D_X(\sigma^{\frac{\kappa}{2}}) = f \cdot s = 1$ . Since  $r_k(1, 1, 0)$  is independent of  $X$ , it must be nonzero.

Assume inductively that for  $1 \leq \ell \leq \kappa - 1$  we have:

(1) for  $z \in \mathbf{A}(\sigma^\perp)$  such that  $2\ell - 1 + \deg z \equiv \deg D_X \pmod{4}$ ,

$$D_X(\sigma^{2\ell-1}z) = r_k(\ell, \ell, 0) D_{X_0}[\ell](z) + \sum_{i=1}^{\ell-1} t_k(\ell, i) \hat{D}_X(\sigma^{2i-1}z),$$

where the coefficient  $t_k(\ell, i)$  depends only on  $\ell, i$ , and  $s^2$ .

(2)  $r_k(\ell, \ell, 0) \neq 0$ .

For  $z \in \mathbf{A}(\sigma^\perp)$ ,

$$\begin{aligned} D_X(\sigma^{2\kappa-1}z) &= \sum_{\ell=1}^{\kappa} \sum_{b=0}^{\lfloor \frac{\kappa-\ell}{2} \rfloor} r_k(\kappa, \ell, b) D_{X_0}[\ell](z x^{\kappa-\ell-2b}) \\ &= r_k(\kappa, \kappa, 0) D_{X_0}[\kappa](z) \\ &\quad + \sum_{\ell=1}^{\kappa-1} \sum_{b=0}^{\lfloor \frac{\kappa-\ell}{2} \rfloor} \left\{ \frac{1}{r_k(\ell, \ell, 0)} (\hat{D}_X(\sigma^{2\ell-1}z x^{\kappa-\ell-2b}) \right. \\ &\quad \left. - \sum_{i=1}^{\ell-1} t_k(\ell, i) \hat{D}_X(\sigma^{2i-1}z x^{\kappa-\ell-2b})) \right\}. \end{aligned}$$

The assumption of simple type implies that  $\hat{D}_X(wx^r) = 2^r \hat{D}_X(w)$  for all  $w \in \mathbf{A}(X)$ ; so the above expression simplifies to

$$r_k(\kappa, \kappa, 0) D_{X_0}[\kappa](z) + \sum_{i=1}^{\kappa-1} t_k(\kappa, i) \hat{D}_X(\sigma^{2i-1}z).$$

To see that  $r_k(\kappa, \kappa, 0) \neq 0$ , let  $X = E(2\kappa + 1) \# 2(k - \kappa) \overline{\mathbf{C}\mathbf{P}}^2$  and let  $\sigma = s - \sum_{i=1}^{2(k-\kappa)} e_i$  where  $s$  is a section of  $E(2\kappa + 1)$ . Then  $\sigma$  is represented by an embedded 2-sphere in  $X$ , and  $\sigma^2 = -(2k + 1)$ . We have seen that the Donaldson series of  $X$  is

$$D_X = \exp(Q/2) \sinh^{2\kappa-1}(f) \prod \cosh(e_i),$$

where  $f$  is the fiber class of  $E(2\kappa + 1)$ . We get

$$D_X(\sigma^{2\kappa-1}) = (2\kappa - 1)! f^{2\kappa-1} (s^{2\kappa-1}) = (2\kappa - 1)!,$$

and also  $D_X(\sigma^{2\ell-1}) = 0$  for  $\ell \leq \kappa$ . It thus follows that  $(2\kappa - 1)! = D_X(\sigma^{2\kappa-1}) = r_k(\kappa, \kappa, 0) D_{X_0}[\kappa](1)$ ; so  $r_k(\kappa, \kappa, 0) \neq 0$ , completing the induction.

For  $\kappa = k$  and any  $z \in \mathbf{A}(\sigma^\perp)$ , we thus get

$$(7) \quad \hat{D}_X(\sigma^{2k-1}z) = r_k(k, k, 0)\hat{D}_{X_0}[k](z) + \sum_{i=1}^{k-1} t_k(k, i)\hat{D}_X(\sigma^{2i-1}z).$$

Our final task in proving Theorem 4.1 for  $\sigma^2 = -(2k + 1)$  is the identification of  $D_{X_0}[k]$ , the relative Donaldson invariant for  $X_0$ , with boundary value  $\zeta^k$ . Let  $\eta$  denote the generator of  $\chi_{SO(3)}(L)$ , the character variety of  $SO(3)$  representations of  $\pi_1(L)$  mod conjugacy. Then we may identify  $\zeta$  with  $\text{ad}(\zeta) = \eta^2$ . By viewing  $SU(2)$  connections over  $X_0$  as  $SO(3)$  connections with  $w_2 = 0$ , we may identify  $D_{X_0}[k] = D_{X_0,0}[\eta^{2k}] = D_{X_0,0}[\eta]$ , since  $\eta^{2k} = \eta^{-1} = \eta$  in  $\chi_{SO(3)}(L)$ .

Consider the calculation of  $\hat{D}_{X,\sigma}(z)$  for  $z \in \mathbf{A}(X_0)$ . Let  $w = z$  in case  $\text{deg } z \equiv \text{deg } D_{X,\sigma} \pmod{4}$ , and  $w \equiv \frac{1}{2}zx$  in case  $\text{deg } z \equiv \text{deg } D_{X,\sigma} + 2 \pmod{4}$ . So  $\hat{D}_{X,\sigma}(z) = D_{X,\sigma}(w)$ . A standard neck-stretching and dimension counting argument show that all connections in the cut-down moduli space  $\mathcal{M}_{X,\sigma} \cap V_w$ , where  $V_w$  is the divisor associated to  $\mu(w)$ , can be obtained by grafting connections in  $\mathcal{M}_{X_0}[\eta]$  to a nontrivial reducible connection over  $N$ , which lies in a moduli space of formal dimension  $\leq -1$ . There is just one such connection, the reducible connection corresponding to the  $SO(3)$  bundle  $\lambda \oplus \mathbf{R}$  over  $N$ . Hence  $\hat{D}_{X,\sigma}(z) = \pm \hat{D}_{X_0,0}[\eta](z) = \pm D_{X_0}[k](z)$ . Theorem 4.1 for  $\sigma^2 = -(2k + 1)$  now follows from (7).

To prove the theorem in the even case  $\sigma^2 = -2k$ , we have a similar induction argument which we start with the two calculations

$$\begin{aligned} \hat{D}_X(z) &= D_{X_0}[0](z), \\ \hat{D}_X(\sigma^2 z) &= s_k(1, 1, 0)D_{X_0}[1](z x^{\frac{1}{2}}), \end{aligned}$$

the second equation by Proposition 4.6. To compute the top term  $\hat{D}_{X_0}[k](z)$ , we once again pass to  $SO(3)$ . This time  $\zeta^k = \eta^{2k} = \eta^0$ , the trivial  $SO(3)$  character; so  $\hat{D}_{X_0}[k](z) = D_{X,\sigma}(z)$ . Theorem 4.1 for  $\sigma^2 = -2k$  follows.

In proving Theorem 4.1 we have not made use of the entire hypothesis of simple type. Rather, we have only needed the assumption that  $D_X(z x^2) = 4 D_X(z)$  for all  $z \in \mathbf{A}(X)$ . Hence by from our formula we obtain

**Lemma 4.7.** *Suppose that  $D_X(z x^2) = 4 D_X(z)$  for all  $z \in \mathbf{A}(X)$  and that  $c \in H_2(X; \mathbf{Z})$  is represented by an embedded sphere of self-intersection  $\leq -2$ . Then*

$$D_{X,c}(z x^2) = 4 D_{X,c}(z)$$

for all  $z \in \mathbf{A}(c^\perp)$ .

Finally, we remark that in case we do not make the assumption that  $X$  has simple type, by keeping track of the powers of  $x$  which occur, our proof still gives a relation.

**Theorem 4.8.** *Let  $X$  be an oriented simply connected 4-manifold, which contains an embedded 2-sphere  $S$  representing an homology class  $\sigma$  with self-intersection  $\sigma^2 \leq -2$ . Then there are constants  $A_{j,b,k}$  and  $B_{j,b,k}$  depending only on  $\sigma^2$ , such that for  $z \in \mathbf{A}(\sigma^\perp)$ ,*

$$D(\sigma^{2k-1}z) = B_{0,0,k}D_\sigma(z) + \sum_{j=1}^{k-1} \sum_{b=0}^{\lfloor \frac{k-j}{2} \rfloor} B_{j,b,k}D(\sigma^{2j-1}x^{k-j-2b}z),$$

if  $\sigma^2 = -(2k + 1)$ ,

and

$$D(\sigma^{2k}z) = A_{0,0,k}D_\sigma(z) + \sum_{j=1}^k \sum_{b=0}^{\lfloor \frac{k-j+1}{2} \rfloor} A_{j,b,k}D(\sigma^{2j-2}x^{k-j-2b+1}z),$$

if  $\sigma^2 = -2k$ ,

and furthermore,  $A_{1,0,k} = 0$ .

In the next section, we shall see that the structure theorem for manifolds of simple type follows from Theorem 4.1 and the blowup formula, Theorem 3.12, by purely formal arguments. Similar arguments should obtain a structure theorem in general from Theorem 4.8 and the general blowup formula [14]. We have not yet worked out the details.

## 5. Structure theory for the Donaldson series

### 5.1. The Donaldson series as a solution to a differential equation

Throughout this section,  $X$  will denote a simply connected 4-manifold of simple type. Suppose that  $c \in H_2(X; \mathbf{Z})$  is represented by an embedded sphere of square  $\leq -2$ . Write  $\mathbf{D}_X = \exp(Q/2)\mathbf{K}_X$  and  $\mathbf{D}_{X,c} = \exp(Q/2)\mathbf{K}_{X,c}$  where for  $\alpha \in H_2(X)$ ,

$$\mathbf{K}_X(\alpha) = K(\exp(\alpha)),$$

which defines  $K \in \mathbf{A}^*(X)$ . Similarly,  $\mathbf{K}_{X,c}(\alpha) = K_{X,c}(\exp(\alpha))$ . We first rewrite our relations of the last section in terms of differential equations for  $\mathbf{K}_X$  and  $\mathbf{K}_{X,c}$ .

For  $u \in H_2(X)$  and  $F \in \mathbf{A}^*(X)$ , the interior product

$$\iota_u F(v) = (\deg(v) + 1)F(uv)$$

gives a derivation. On the formal power series  $\mathbf{F}$  on  $H_2(X)$  defined by  $\mathbf{F}(\alpha) = F(\exp(\alpha))$  this induces

$$\partial_u \mathbf{F}(\alpha) = F(u \exp(\alpha)),$$

which is just the formal derivative of  $\mathbf{F}$  in the direction  $u$ . Similarly, for higher order derivatives,  $\partial_u^k \mathbf{F}(\alpha) = F(u^k \exp(\alpha))$ . Also, note that the linearity of  $F$  implies that  $\partial_{u+v} \mathbf{F} = \partial_u \mathbf{F} + \partial_v \mathbf{F}$ .

An induction argument shows that

$$(8) \quad \partial_u^{2k} \exp(Q/2) = \exp(Q/2) \sum_{t=0}^k (u \cdot u)^{k-t} \tilde{u}^{2t} \binom{2k}{2t} \frac{(2k-2t)!}{2^{k-t}(k-t)!}$$

and

$$(9) \quad \begin{aligned} \partial_u^{2k+1} \exp(Q/2) &= \exp(Q/2) \sum_{t=0}^k (u \cdot u)^{k-t} \tilde{u}^{2t+1} \binom{2k+1}{2t+1} \frac{(2k-2t)!}{2^{k-t}(k-t)!}, \end{aligned}$$

where  $\tilde{u}$  is the dual form of  $u \in H_2(X)$  with respect to the intersection form  $Q$ . For  $\kappa \in H_2(X)$  we have

$$(10) \quad \partial_u(e^\kappa) = (\kappa \cdot u)e^\kappa.$$

Using equations (8) and (9) we can restate the relations given in Theorem 4.1.

**Theorem 5.1.** *Suppose  $u \in H_2(X; \mathbf{Z})$  is represented by an embedded sphere. Let  $k \geq 1$  be an integer. If  $u^2 = -2k$ , then there are constants  $a_{i,k}$  depending only on  $u^2$  such that on  $\langle u \rangle^\perp$ ,*

$$(\partial_u^{2k} + a_{1,k} \partial_u^{2k-2} + \dots + a_{k-1,k} \partial_u^2 + a_{k,k}) \mathbf{K}_X + a_{k+1,k} \mathbf{K}_{X,u} = 0.$$

*If  $u^2 = -(2k+1)$ , then there are constants  $b_{i,k}$  depending only on  $u^2$  such that on  $\langle u \rangle^\perp$ ,*

$$(\partial_u^{2k-1} + b_{1,k} \partial_u^{2k-3} + \dots + b_{k-1,k} \partial_u) \mathbf{K}_X + b_{k,k} \mathbf{K}_{X,u} = 0.$$

Note that the equations in Theorem 5.1 involve both  $SU(2)$ -invariants  $\mathbf{K}_X$  and  $SO(3)$ -invariants  $\mathbf{K}_{X,u}$ . We first use the blowup formula (Theorem 3.12) and our computations for the elliptic surfaces  $E(2k)$  (Theorem 3.1) to determine differential equations, which do not involve the  $\mathbf{K}_{X,u}$ , and are valid on all of  $\mathbf{A}(X)$ . At the end of this section we will return to Theorem 5.1 to show that the  $SO(3)$ -Donaldson invariants are determined by the  $SU(2)$ -Donaldson invariants.

It is convenient to define the differential operators

$$\Delta_{u,k}^e = \partial_u(\partial_u^2 - 2^2)(\partial_u^2 - 4^2) \cdots (\partial_u^2 - (2k-2)^2)$$

and

$$\Delta_{u,k}^{\circ} = (\partial_u^2 - 1)(\partial_u^2 - 3^2) \cdots (\partial_u^2 - (2k - 1)^2).$$

Suppose  $u \in H_2(X)$  and let  $e \in H_2(X \# \overline{\mathbf{CP}}^2; \mathbf{Z})$  be the exceptional class. Then the following differentiation formulas can be verified by induction on  $k$  using the fact that  $\partial_{u+e} = \partial_u + \partial_e$ :

$$(11) \quad \Delta_{u+e,k}^e(\mathbf{K}_X \cosh(e)) = \cosh(e)\partial_u \Delta_{u,k-1}^{\circ}(\mathbf{K}_X) - (2k - 1) \sinh(e)\Delta_{u,k-1}^{\circ}(\mathbf{K}_X),$$

$$(12) \quad \begin{aligned} \Delta_{u+e,k}^{\circ}(\mathbf{K}_X \cosh(e)) &= \cosh(e)\partial_u \Delta_{u,k}^e(\mathbf{K}_X) - 2k \sinh(e)\Delta_{u,k}^e(\mathbf{K}_X). \end{aligned}$$

**Theorem 5.2.** *Suppose  $u \in H_2(X; \mathbf{Z})$  is represented by an embedded sphere. If  $u^2 = -2k$ ,  $k \geq 1$ , then on  $\langle u \rangle^{\perp}$ ,*

$$\Delta_{u,k}^e(\mathbf{K}_X) = 0.$$

*Proof.* We first show that there are constants  $d_{i,k}$  depending only on  $u^2$  such that on  $\langle u \rangle^{\perp}$

$$(13) \quad (\partial_u^{2k-1} + d_{1,k}\partial_u^{2k-3} + \cdots + d_{k-1,k}\partial_u)\mathbf{K}_X = 0.$$

To do this, consider  $u + e \in H_2(X \# \overline{\mathbf{CP}}^2; \mathbf{Z})$ . It is represented by a sphere which has square  $-(2k + 1)$ . By Theorem 5.1, on  $\langle u + e \rangle^{\perp}$

$$\begin{aligned} 0 &= (\partial_{u+e}^{2k-1} + b_{1,k}\partial_{u+e}^{2k-3} + \cdots + b_{k-1,k}\partial_{u+e})\mathbf{K}_{X \# \overline{\mathbf{CP}}^2} + b_{k,k}\mathbf{K}_{X \# \overline{\mathbf{CP}}^2, u+e} \\ &= (\partial_{u+e}^{2k-1} + b_{1,k}\partial_{u+e}^{2k-3} + \cdots + b_{k-1,k}\partial_{u+e})\mathbf{K}_X \cosh(e) \\ &\quad - b_{k,k}\mathbf{K}_{X,u} \sinh(e). \end{aligned}$$

For the last equality we use the fact that

$$\mathbf{K}_{X \# \overline{\mathbf{CP}}^2, u+e} = -\mathbf{K}_{X,u} \sinh(e),$$

which follows from Theorem 3.12. Differentiating and using  $\partial_{u+e} = \partial_u + \partial_e$ , we get constants  $d_{i,k}$  and  $d'_{i,k}$  with

$$\begin{aligned} &\cosh(e)\{\partial_u^{2k-1} + d_{1,k}\partial_u^{2k-3} + \cdots + d_{k-1,k}\partial_u\}(\mathbf{K}_X) \\ &\quad - \sinh(e)\{((2k - 1)\partial_u^{2k-2} + d'_{1,k}\partial_u^{2k-4} + \cdots + d'_{k-1,k})\}(\mathbf{K}_X) \\ &\quad \quad \quad + d'_{k,k}\mathbf{K}_{X,u}\} = 0. \end{aligned}$$

Now  $\langle u + e \rangle^{\perp} = \{y + (u \cdot y)e \mid y \in H_2(X)\}$ . Note that

$$\begin{aligned} \cosh(e)((y + (u \cdot y)e)^m) &= \cosh(u)(y^m), \\ \sinh(e)((y + (u \cdot y)e)^m) &= -\sinh(u)(y^m). \end{aligned}$$

Hence restricting the last equation to  $\langle u + e \rangle^\perp$  we get

$$(14) \quad \cosh(u) \{ \partial_u^{2k-1} + d_{1,k} \partial_u^{2k-3} + \cdots + d_{k-1,k} \partial_u \} (\mathbf{K}_X) \\ + \sinh(u) \{ ((2k-1) \partial_u^{2k-2} + d'_{1,k} \partial_u^{2k-4} + \cdots + d'_{k-1,k}) (\mathbf{K}_X) \\ + d'_{k,k} \mathbf{K}_{X,u} \} = 0.$$

Restricting to  $\langle u \rangle^\perp$  thus gives the desired equation.

To determine the universal constants  $d_{i,k}$  we substitute

$$\mathbf{K}_{E(2k)} = \sinh^{2k-2}(f) \\ = \frac{1}{2^{2k-3}} \left\{ \sum_{r=1}^{k-1} (-1)^{k-1+r} \binom{2k-2}{k-1-r} \cosh(2rf) \right. \\ \left. + (-1)^{k-1} \frac{1}{2} \binom{2k-2}{k-1} \right\}.$$

into (13) where  $u$  is a section of  $E(2k)$ ; so  $u^2 = -2k$  and  $u \cdot f = 1$ . (Note that if  $v = u + 2kf$  then  $v \cdot u = 0$  and  $v \cdot f \neq 0$ ; so we are not applying the equation to a trivial situation.) For each  $1 \leq r \leq k-1$  we get the equation

$$(15) \quad \{ (2r)^{2k-1} + \sum_{t=1}^{k-1} d_{k-t,k} (2r)^{2t-1} \} \sinh(2rf) = 0.$$

The characteristic equation of (13) is given by

$$(16) \quad z^{2k-1} + \sum_{t=1}^{k-1} d_{k-t,k} z^{2t-1} = 0,$$

and from (15) it follows that  $z = 0, \pm 2, \pm 4, \pm 2(k-1)$  are the characteristic roots, which are precisely the characteristic roots of  $\Delta_{u,k}^e$ .

**Lemma 5.3.** *Suppose that  $u \in H_2(X; \mathbf{Z})$  is represented by an embedded sphere. If  $u^2 = -2k$ ,  $k > 0$ , then there are constants  $d_{i,k}$  and  $d'_{i,k}$  depending only on  $u^2$  such that*

$$\cosh(u) \{ \partial_u^{2k-1} + d_{1,k} \partial_u^{2k-3} + \cdots + d_{k-1,k} \partial_u \} (\mathbf{K}_X) \\ + \sinh(u) \{ ((2k-1) \partial_u^{2k-2} + d'_{1,k} \partial_u^{2k-4} + \cdots + d'_{k-1,k}) (\mathbf{K}_X) \\ + d'_{k,k} \mathbf{K}_{X,u} \} = 0$$

holds on all  $H_2(X)$ . If  $u^2 = -(2k+1)$ ,  $k > 0$ , then there are constants  $h_{i,k}$  and  $h'_{i,k}$  depending only on  $u^2$  such that

$$\begin{aligned} & \cosh(u) \{ \partial_u^{2k} + h_{1,k} \partial_u^{2k-2} + \cdots + h_{k-1,k} \partial_u^2 + h_{k,k} \} (\mathbf{K}_X) \\ & + \sinh(u) \{ ((2k) \partial_u^{2k-1} + h'_{1,k} \partial_u^{2k-3} + \cdots + h'_{k-1,k} \partial_u) (\mathbf{K}_X) \\ & + h'_{k,k} \mathbf{K}_{X,u} \} = 0 \end{aligned}$$

holds on all  $H_2(X)$ .

*Proof.* The  $u^2 = -2k$  case is (15) above. The odd case is proved similarly.

**Theorem 5.4.** Suppose  $u \in H_2(X; \mathbf{Z})$  is represented by an embedded sphere of square  $u^2 = -(2k-1) \leq -3$ . Then on all of  $H_2(X)$ ,

$$\cosh(u) \partial_u \Delta_{u,k-1}^\circ (\mathbf{K}_X) + (2k-1) \sinh(u) \Delta_{u,k-1}^\circ (\mathbf{K}_X) = 0.$$

In particular,

$$\partial_u \Delta_{u,k-1}^\circ (\mathbf{K}_X) = 0$$

on  $\langle u \rangle^\perp$ .

*Proof.* Again consider  $u+e \in H_2(X \# \overline{\mathbf{CP}}^2; \mathbf{Z})$ . Then  $(u+e)^2 = -2k$  and  $u+e$  is represented by a sphere. Thus, by Theorem 5.2, on  $\langle u+e \rangle^\perp$ ,

$$0 = \Delta_{u+e,k}^e (\mathbf{K}_{X \# \overline{\mathbf{CP}}^2}) = \Delta_{u+e,k}^e (\mathbf{K}_X \cosh(e)).$$

Now apply the differentiation formula (12) and restrict to  $H_2(X)$ , as in the proof of Theorem 5.2, to obtain the desired differential equation.

**Corollary 5.5.** Suppose that  $u \in H_2(X; \mathbf{Z})$  is represented by an embedded sphere of square  $u^2 = -(2k-1) \leq -1$ . Then on all of  $H_2(X)$ ,  $\mathbf{K}_X$  satisfies the constant coefficient homogeneous linear ordinary differential equation

$$\Delta_{u,k}^\circ (\mathbf{K}_X) = 0.$$

*Proof.* If  $u^2 = -1$ , the result follows directly from Theorem 3.12. Suppose that  $u^2 \leq -3$ . By Theorem 5.4,

$$\cosh(u) \partial_u \Delta_{u,k-1}^\circ (\mathbf{K}_X) + (2k-1) \sinh(u) \Delta_{u,k-1}^\circ (\mathbf{K}_X) = 0$$

on all of  $H_2(X)$ . Differentiate this equation to get

$$\begin{aligned} 0 &= \partial_u (\cosh(u) \partial_u \Delta_{u,k-1}^\circ (\mathbf{K}_X) + (2k-1) \sinh(u) \Delta_{u,k-1}^\circ (\mathbf{K}_X)) \\ &= \cosh(u) (\partial_u^2 - (2k-1)^2) \Delta_{u,k-1}^\circ (\mathbf{K}_X) = \cosh(u) \Delta_{u,k}^\circ (\mathbf{K}_X), \end{aligned}$$

and the result follows.



**Corollary 5.6.** *Suppose  $u \in H_2(X; \mathbf{Z})$  is represented by an embedded sphere. If  $u^2 = -2k \leq -2$ , then on all of  $H_2(X)$ ,*

$$\Delta_{u,k+1}^e(\mathbf{K}_X) = 0.$$

*Proof.* Again consider  $u + e \in H_2(X \# \overline{\mathbf{C}\mathbf{P}}^2; \mathbf{Z})$ . Then  $(u + e)^2 = -(2k + 1)$ , and  $u + e$  is represented by a sphere. Thus

$$\begin{aligned} 0 &= \Delta_{u+e,k+1}^o \mathbf{K}_{X \# \overline{\mathbf{C}\mathbf{P}}^2} = \Delta_{u+e,k+1}^o(\mathbf{K}_X \cosh(e)) \\ &= \cosh(e) \partial_u \Delta_{u,k+1}^e(\mathbf{K}_X) - 2(k + 1) \sinh(e) \Delta_{u,k+1}^e(\mathbf{K}_X) \end{aligned}$$

on all of  $H_2(X \# \overline{\mathbf{C}\mathbf{P}}^2)$ . Since  $\cosh(e)$  and  $\sinh(e)$  are linearly independent functions on  $H_2(X \# \overline{\mathbf{C}\mathbf{P}}^2)$ ,

$$\Delta_{u,k+1}^e(\mathbf{K}_X) = 0.$$

We shall also need the following stronger result.

**Theorem 5.7.** *Suppose  $u \in H_2(X; \mathbf{Z})$  is represented by an embedded sphere of square  $u^2 = -2k \leq 2$ . Then on all of  $H_2(X)$ ,*

$$\cosh(u) \Delta_{u,k}^e(\partial_u^2 + 2k)(\mathbf{K}_X) + (2k + 1) \sinh(u) \partial_u \Delta_{u,k}^e(\mathbf{K}_X) = 0.$$

*Proof.* The proof is similar to that of Theorem 5.4. Consider  $u + e \in H_2(X \# \overline{\mathbf{C}\mathbf{P}}^2; \mathbf{Z})$ . It is represented by a sphere of square  $-(2k + 1)$ . By Theorem 5.4, on  $\langle u + e \rangle^\perp$

$$0 = \partial_{u+e} \Delta_{u+e,k}^o \mathbf{K}_{X \# \overline{\mathbf{C}\mathbf{P}}^2} = \partial_{u+e} \Delta_{u+e,k}^o(\mathbf{K}_X \cosh(e)).$$

Apply the differentiation formula (13), and project back to  $H_2(X)$ , as in the proof of Theorem 5.4, to obtain the desired differential equation.

Now, for arbitrary  $u \in H_2(X; \mathbf{Z})$ , we determine differential equations with respect to  $\partial_u$  that  $\mathbf{K}_X$  satisfy. For this, the following differentiation formulas, again proved by induction on  $k$  using  $\partial_{u-2e} = \partial_u - 2\partial_e$ , are essential:

$$\begin{aligned} \Delta_{u-2e,k}^e(\mathbf{K}_X \cosh(e)) &= \cosh(e) \Delta_{u,k-1}^e(\partial_u^2 + 4k(k - 1))(\mathbf{K}_X) \\ (17) \qquad \qquad \qquad &+ 2(2k - 1) \sinh(e) \partial_u \Delta_{u,k-1}^e(\mathbf{K}_X), \end{aligned}$$

$$\begin{aligned} \Delta_{u-2e,k}^o(\mathbf{K}_X \cosh(e)) &= \cosh(e) \Delta_{u,k-1}^o(\partial_u^2 + (2k + 1)(2k - 1))(\mathbf{K}_X) \\ (18) \qquad \qquad \qquad &+ 4k \sinh(e) \partial_u \Delta_{u,k-1}^o(\mathbf{K}_X). \end{aligned}$$

**Theorem 5.8.** *Suppose  $u \in H_2(X; \mathbf{Z})$  can be represented by an immersed sphere with  $p_u$  positive double points (and an arbitrary number*

of negative double points). If  $u^2 = 2k + 1$ , then on  $H_2(X)$ ,  $\mathbf{K}_X$  satisfies the differential equation

$$\Delta_{u, p_u - k}^{\circ}(\mathbf{K}_X) = 0.$$

If  $u^2 = 2k$ , then on  $H_2(X)$ ,  $\mathbf{K}_X$  satisfies

$$\Delta_{u, p_u - k + 1}^e(\mathbf{K}_X) = 0.$$

*Proof.* To begin, suppose that  $u \in H_2(X; \mathbf{Z})$  with  $u^2 = 2k + 1$  is represented by an immersed sphere with  $p_u$  positive double points and  $n_u$  negative double points. Then in  $\hat{X} = X \#_{p_u} \overline{\mathbf{C}\mathbf{P}}^2 \#_{n_u} \overline{\mathbf{C}\mathbf{P}}^2$  the homology class  $\hat{u} = u - 2e_1 - \dots - 2e_{p_u}$  is represented by an embedded sphere with  $\hat{u}^2 = 2k + 1 - 4p_u = -(2(2p_u - k) - 1)$ . If a simply connected 4-manifold contains an embedded essential sphere of nonnegative self-intersection, then all the Donaldson invariants of the manifold vanish [28]. Thus  $\hat{u}^2 < 0$ . By Corollary 5.5, we have

$$\Delta_{\hat{u}, 2p_u - k}^{\circ}(\mathbf{K}_{\hat{X}}) = 0.$$

Consider the differentiation formula (18). In this formula, note that if

$$\Delta_{u - 2e, k}^{\circ}(\mathbf{K}_X \cosh(e)) = 0$$

everywhere, then on  $H_2(X)$  one obtains the two differential equations,

$$\Delta_{u, k - 1}^{\circ}(\partial_u^2 + (2k + 1)(2k - 1))(\mathbf{K}_X) = 0$$

and

$$\partial_u \Delta_{u, k - 1}^{\circ}(\mathbf{K}_X) = 0.$$

The common solutions satisfy

$$\Delta_{u, k - 1}^{\circ} \mathbf{K}_X = 0.$$

Thus for each of  $e_1, \dots, e_{p_u}$ , the differential equation which results from applying (18) has two fewer characteristic roots. Thus,  $p_u$  applications of this differentiation formula to

$$\Delta_{\hat{u}, 2p_u - k}^{\circ}(\mathbf{K}_{\hat{X}}) = 0$$

yields the desired result. A similar argument applies when  $u^2 = 2k$ .

Note that if  $\mathbf{D}_X \neq 0$ , this theorem implies that  $p_u > \frac{1}{2}(u^2 - 1)$  if  $u^2$  is odd and  $p_u > \frac{1}{2}(u^2 - 2)$  if  $u^2$  is even. We shall improve this considerably in Theorem 5.10 below.

## 5.2. The structure of the Donaldson series

In this section we shall show how the constant coefficient linear homogeneous ordinary differential equations given by Theorem 5.8 determine the qualitative structure of the  $SU(2)$  Donaldson series.

**Theorem 5.9.** *If  $X$  is a simply connected 4-manifold of simple type, then there exist finitely many homology classes  $\kappa_1, \dots, \kappa_p \in H_2(X, \mathbf{Z})$  and nonzero rational numbers  $a_1, \dots, a_p$  such that*

$$\mathbf{D}_X = \exp(Q/2) \sum_{s=1}^p a_s e^{\kappa_s}$$

as analytic functions on  $H_2(X)$ . Each of the classes  $\kappa_s$  is characteristic, i.e., an integral lift of  $w_2(X)$ , and the collection  $\{\kappa_s\}$  is a diffeomorphism invariant of  $X$ .

Theorem 5.9 was first proved by Kronheimer and Mrowka [24] through their detailed study of singular connections. The classes  $\kappa_s$  are called *basic classes*. Note that since  $\mathbf{D}_X$  is an even function if  $b_X^+ \equiv 3 \pmod{4}$ , and an odd function if  $b_X^+ \equiv 1 \pmod{4}$ , the nonzero basic classes come in pairs,  $\kappa_s$  and  $-\kappa_s$ . Thus we can rewrite  $\mathbf{D}_X$  as follows. If  $b_X^+ \equiv 3 \pmod{4}$ , then

$$\mathbf{D}_X = \exp(Q/2) \sum_{s=1}^r b_s \cosh(\kappa_s),$$

and if  $b_X^+ \equiv 1 \pmod{4}$  then

$$\mathbf{D}_X = \exp(Q/2) \sum_{s=1}^r b_s \sinh(\kappa_s).$$

Beware that there are ambiguities when writing  $\mathbf{D}_X$  this way. We will adopt the convention which dictates that both  $\kappa_s$  and  $-\kappa_s$  do not occur as an argument of these hyperbolic trigonometric functions. If we replace the chosen  $\kappa_s$  by  $-\kappa_s$  when  $b_X^+ \equiv 3 \pmod{4}$ , then the coefficient  $b_s$  remains unchanged. However, when  $b_X^+ \equiv 1 \pmod{4}$  the sign of the coefficient  $b_s$  changes.

*Proof of Theorem 5.9.* Write  $\mathbf{D}_X = \exp(Q/2)\mathbf{K}_X$ . Let  $u_1, \dots, u_b \in H_2(X)$  be a basis represented by immersed spheres and ordered so that  $u_t^2 = 2k_t + 1$  for  $t = 1, \dots, \ell$  and  $u_t^2 = 2k_t$  for  $t = \ell + 1, \dots, b$ . Using the blowup formula, Theorem 3.12, we may assume that no  $u_t$  is represented by a sphere of self-intersection  $-1$ . Let  $u_1^*, \dots, u_b^* \in H_2(X; \mathbf{Z})$  be a dual basis satisfying  $u_i^* \cdot u_j = \delta_{ij}$ . By Theorem 5.8,  $\mathbf{K}_X$  satisfies the system of constant coefficient linear differential equations given by

$$\begin{aligned} \Delta_{u_t, p_{u_t} - k}^{\circ}(\mathbf{K}_X) &= 0 & (t = 1, \dots, \ell), \\ \Delta_{u_t, p_{u_t} - k + 1}^{\circ}(\mathbf{K}_X) &= 0 & (t = \ell + 1, \dots, b). \end{aligned}$$

Note that each equation is an ordinary differential equation in a single variable while the other variables are held fixed. The roots of the corresponding characteristic polynomials are distinct and lie in sets of inte-

gers  $N_1, \dots, N_b$  of orders  $p_1, \dots, p_b$ , respectively. More specifically,  $N_t = \{\pm 1, \pm 3, \pm 2(p_{u_t} - k_t) - 1\}$  for  $t = 1, \dots, \ell$ ; and  $N_t = \{0, \pm 2, \pm 2(p_{u_t} - k_t)\}$  for  $t = \ell + 1, \dots, b$ . For each  $t = 1, \dots, b$ , let us denote the characteristic roots lying in  $N_t$  by  $r_{t,j_t}$  for  $j_t = 1, \dots, p_t$ . Solving the first equation gives

$$\mathbf{K}_X = \sum_{j_1=1}^{p_1} \Phi_{j_1} e^{r_{1,j_1} u_1^*},$$

where each  $\partial_{u_1} \Phi_{j_1} = 0$ . Substitute this expression for  $\mathbf{K}_X$  into the second equation. The fact that  $u_1^* \cdot u_2 = 0$  implies that each  $\Phi_{j_1}$  satisfies the second equation. Solving gives

$$\Phi_{j_1} = \sum_{j_2=1}^{p_2} \Phi_{j_1,j_2} e^{r_{2,j_2} u_2^*},$$

where each  $\partial_{u_2} \Phi_{j_1,j_2} = 0$ . Also since  $\partial_{u_1} \Phi_{j_1} = 0$ , each  $\partial_{u_1} \Phi_{j_1,j_2} = 0$  as well. Hence

$$\mathbf{K}_X = \sum_{j_1=1}^{p_1} \sum_{j_2=1}^{p_2} \Phi_{j_1,j_2} e^{r_{1,j_1} u_1^* + r_{2,j_2} u_2^*}.$$

Inductively, we get

$$\mathbf{K}_X = \sum_{j_1=1}^{p_1} \cdots \sum_{j_b=1}^{p_b} \Phi_{j_1, \dots, j_b} e^{r_{1,j_1} u_1^* + \cdots + r_{b,j_b} u_b^*},$$

where the coefficients  $\Phi_{j_1, \dots, j_b}$  satisfy  $\partial_{u_t} \Phi_{j_1, \dots, j_b} = 0$  for  $t = 1, \dots, b$ . I.e., the coefficients are constant; so we have

$$(19) \quad \mathbf{K}_X = \sum_{s=1}^p a_s e^{\kappa_s}$$

as advertised. Since the functions  $e^\alpha : H_2(X) \rightarrow \mathbf{R}$  are linearly independent for distinct  $\alpha$ , it follows that the set of classes  $\kappa_s$  corresponding to  $a_s \neq 0$  is a diffeomorphism invariant of  $X$ .

To see that the coefficients  $a_s$  must be rational, assume, e.g. that  $b_X^+ \equiv 3 \pmod{4}$ ; so as above

$$\mathbf{D}_X = \exp(Q/2) \sum_{s=1}^p b_s \cosh(\kappa_s),$$

where  $b_s = 2a_s$ . Consider any  $u \in H_2(X; \mathbf{Z})$  and nonnegative integer  $d$  with  $\hat{D}(u^d) \neq 0$ , an integer. Then

$$\hat{D}(u^d) = d! \sum_{s=1}^p b_s \sum_{j=0}^{\frac{d}{2}} \frac{(u^2)^j}{2^j j!} \frac{(\kappa_s \cdot u)^{d-2j}}{(d-2j)!} = \sum_{s=1}^p \eta_s(u, d) b_s,$$

where the coefficients  $\eta_s(u, d)$  are rational numbers. If we can find  $p$  such pairs  $(u_i, d_i)$  such that the  $p \times p$  matrix of coefficients  $\{\eta_s(u_i, d_i)\}$  is nonsingular, then it follows that the  $b_s$  and hence the  $a_s$  are rational. However, if we cannot find  $p$  such pairs, then choose among all such pairs so that  $\{\eta_s(u_i, d_i)\}$  has maximum rank  $r$ , and renumber the pairs and the  $\kappa_s$ 's so that the upper lefthand  $r \times r$  submatrix  $A_r$  is nonsingular. Then for  $j, t > r$ , we can solve

$$\eta_t(u_j, d_j) = \sum_{s=1}^r \frac{\zeta_s}{\det A} \eta_s(u_i, d_i),$$

where  $\zeta_t$  is a subdeterminant of the first  $r$  rows of  $\{\eta_s(u_i, d_i)\}$ . This means that we can express

$$D_X = \exp(Q/2) \sum_{s=1}^r b'_s \cosh(\kappa_s),$$

contradicting the diffeomorphism invariance of the set of basic classes.

Since each  $r_{t,jt} \equiv u_t^2 \pmod{2}$ , each of the basic classes  $r_{1,j1} u_1^* + \dots + r_{b,jb} u_b^*$  is characteristic.

**5.3. Generalized adjunction formulas**

Let  $X$  be a simply connected 4-manifold of simple type with a nontrivial Donaldson invariant. According to our comments above, its basic classes come in pairs  $\{\kappa_s, -\kappa_s\}$ , and

$$D_X = \exp(Q/2) \left\{ \sum_{s=1}^r a_s e^{\kappa_s} + (-1)^{\frac{1+b^+}{2} X} \sum_{s=1}^r a_s e^{-\kappa_s} \right\}.$$

**Theorem 5.10.** *Let  $X$  be a simply connected 4-manifold of simple type and let  $\{\kappa_s\}$  be the set of basic classes as above. If  $u \in H_2(X; \mathbf{Z})$  is represented by an immersed 2-sphere with  $p \geq 1$  positive double points, then for each  $s$*

$$(20) \quad 2p - 2 \geq u^2 + |\kappa_s \cdot u|.$$

**Theorem 5.11.** *Let  $X$  be a simply connected 4-manifold of simple type with basic classes  $\{\kappa_s\}$  as above. If the nontrivial class  $u \in H_2(X; \mathbf{Z})$  is represented by an immersed 2-sphere with no positive double points, then let*

$$\{\kappa_s \mid s = 1, \dots, 2m\}$$

*be the collection of basic classes which violate the inequality (20), and thus  $\kappa_s \cdot u = \pm u^2$  for each such  $\kappa_s$ . Order these classes so that  $\kappa_s \cdot u =$*

$-u^2 (> 0)$  for  $s = 1, \dots, m$ . Then

$$(21) \quad \sum_{s=1}^m a_s e^{\kappa_s + u} - (-1)^{\frac{1+b^+}{2}X} \sum_{s=1}^m a_s e^{-\kappa_s - u} = 0.$$

The exceptional cases in Theorem 5.11 do indeed occur; e.g. when  $u$  is the sum of  $(\pm)$  exceptional divisors in  $X \#_k \overline{\mathbf{CP}}^2$ . It could be that these are precisely the exceptional cases.

*Proofs of (5.10) and (5.11).* We have

$$\mathbf{K}_X = \sum_{s=1}^p a_s e^{\kappa_s}.$$

To begin, suppose that  $u \in H_2(X; \mathbf{Z})$  is represented by an embedded sphere with  $u^2 = -(2k - 1) \leq -3$ . Applying Corollary 5.5 to  $\mathbf{K}_X$  yields

$$\sum_{s=1}^p a_s \Delta_{u,k}^{\circ}(e^{\kappa_s}) = 0.$$

Now the  $\kappa_s$  are characteristic; so  $\kappa_s \cdot u \neq 0$ , and  $\partial_u^n(e^{\kappa_s}) = (\kappa_s \cdot u)^n e^{\kappa_s}$ . Thus for each  $s$ ,  $\kappa_s \cdot u$  is a characteristic root of  $\Delta_{u,k}^{\circ}$ , hence  $\kappa_s \cdot u = \pm 1, \pm 3, \dots, \pm(2k - 1)$ . Inequality (20) is satisfied unless  $\kappa_s \cdot u = \pm(2k - 1)$ . Apply Theorem to  $\mathbf{K}_X$  to obtain

$$(22) \quad \cosh(u) \{ \sum_{s=1}^p a_s (\kappa_s \cdot u) \Delta_{u,k-1}^{\circ}(e^{\kappa_s}) \} + (2k - 1) \sinh(u) \{ \sum_{s=1}^p a_s \Delta_{u,k-1}^{\circ}(e^{\kappa_s}) \} = 0.$$

The left-hand side of (23) is 0 for all those  $\kappa_s$  with  $\kappa_s \cdot u \neq \pm(2k - 1)$ , for such  $\kappa_s \cdot u$  are characteristic roots of  $\Delta_{u,k-1}^{\circ}$ . To deal with those  $s$  with  $\kappa_s \cdot u = \pm(2k - 1)$  we reorder the basic classes as in the statement of the theorem so that  $\kappa_1, \dots, \kappa_m$  are those satisfying  $\kappa_s \cdot u = +(2k - 1)$  and then  $-\kappa_1, \dots, -\kappa_m$  are the others. Hence (23) becomes

$$\begin{aligned} 0 &= (2k - 1) \{ \cosh(u) + \sinh(u) \} \sum_{s=1}^m a_s \Delta_{u,k-1}^{\circ}(e^{\kappa_s}) \\ &\quad + (2k - 1) \{ -\cosh(u) + \sinh(u) \} \delta \sum_{s=1}^m a_s \Delta_{u,k-1}^{\circ}(e^{-\kappa_s}) \\ &= (2k - 1) q(2k - 1) \left\{ \sum_{s=1}^m a_s e^{\kappa_s + u} - \delta \sum_{s=1}^m a_s e^{-\kappa_s - u} \right\} \end{aligned}$$

where  $\delta = (-1)^{\frac{1+b^+}{2}X}$ , and  $q$  is the characteristic polynomial of  $\Delta_{u,k-1}^{\circ}$ . (Note that  $q$  is an even function.) Since  $2k - 1$  is not a characteristic

root of  $\Delta_{u,k-1}^o$ ,  $q(2k - 1) \neq 0$ , it follows that

$$\sum_{s=1}^m a_s e^{\kappa_s + u} - \delta \sum_{s=1}^m a_s e^{-\kappa_s - u} = 0.$$

Remark that the case left out, namely that  $u$  is represented by an embedded 2-sphere of square  $-1$ , is dealt with handily by the blowup formula, Theorem 3.12.

Similarly, if  $u^2 = -2k < 0$ , apply Corollary 5.6 to  $\mathbf{K}_X$  to get

$$\sum_{s=1}^p a_s \Delta_{u,k+1}^e(e^{\kappa_s}) = 0,$$

so that  $\kappa_s \cdot u$  is a characteristic root of  $\Delta_{u,k+1}^e$ , hence  $\kappa_s \cdot u = 0, \pm 2, \pm 4, \dots, \pm 2k$ . Again, (20) is satisfied unless  $k_s \cdot u = \pm 2k$ . As above, we use the stronger differential equation given by Theorem 5.7 to obtain

$$\sum_{s=1}^m a_s e^{\kappa_s + u} - \delta \sum_{s=1}^m a_s e^{-\kappa_s - u} = 0,$$

where  $k_s \cdot u = -2k$  for  $s = 1, \dots, m$ .

The upshot of all this is that unless

$$\sum_{s=1}^m a_s e^{\kappa_s + u} - \delta \sum_{s=1}^m a_s e^{-\kappa_s - u} = 0,$$

we have that for all  $s$ ,

$$-2 \geq u^2 + |\kappa_s \cdot u|.$$

More generally, suppose  $u \in H_2(X; \mathbf{Z})$  is represented by an immersed sphere with  $p > 0$  positive double points and  $n$  negative double points. Then, in  $\hat{X} = X \# p \overline{\mathbf{C}\mathbf{P}}^2 \# n \overline{\mathbf{C}\mathbf{P}}^2$  the homology class  $\hat{u} = u - 2e_1 - \dots - 2e_p$  is represented by an embedded sphere and  $\hat{u}^2 = u^2 - 4p$ , which is negative if  $\mathbf{D}_X \neq 0$ . The basic classes of  $\hat{X}$  are of the form

$$\hat{\kappa}_s = \kappa_s \pm e_1 \cdots \pm e_{p+n}.$$

Let  $\{\pm 1\}^{p+n} = \{\xi(1), \dots, \xi(2^{p+n})\}$ . If there is an exceptional case of the theorem occurring here, then there are basic classes  $\hat{\kappa}_{s,n} = \kappa_s + \xi(n)_1 e_1 + \dots + \xi(n)_{p+n} e_{p+n}$  such that

$$(23) \quad \sum a_{s,n} e^{\hat{\kappa}_{s,n} + \hat{u}} - \delta \sum a_{s,n} e^{-\hat{\kappa}_{s,n} - \hat{u}} = 0.$$

However,

$$\hat{\kappa}_{s,n} + \hat{u} = \kappa_s + u + \sum_i (\xi(n)_i - 2) e_i,$$

so classes  $\{\hat{\kappa}_{s,n} + \hat{u}, -\hat{\kappa}_{s,m} - \hat{u}\}$  are distinct. Hence the exponentials occurring in (23) are linearly independent, a contradiction. Thus there are no exceptional cases and

$$-2 \geq \hat{u}^2 + \max_s \{ |(\kappa_s \pm e_1 \cdots \pm e_{p+n}) \cdot (u - 2e_1 - \cdots - 2e_p)| \}.$$

This also satisfies the inequality

$$2p - 2 \geq u^2 + \max_s |\kappa_s \cdot u|.$$

Finally, if  $p = 0$ , and we are in the exceptional case, then we obtain the analogue of (21) for  $\hat{u}$ , and the exceptional classes factor out leaving the desired equation.

Suppose that  $X$  is a simply connected compact complex algebraic surface, and let  $\kappa_X$  denote its canonical class. If  $C$  is an algebraic curve representing an homology class  $c \in H_2(X; \mathbf{Z})$ , then the adjunction formula gives

$$2g(C) - 2 = c^2 + \kappa_X \cdot c,$$

where  $g(C)$  is the genus of  $C$ . There are currently no known examples where  $\kappa_X$  is not also a basic class for  $X$ . This problem has been studied by R. Brussee [6]. At any rate, suppose that it is the case where some  $\kappa_s = \kappa_X$ , and suppose that there is a smoothly immersed 2-sphere  $T$  in  $X$ , with  $p$  positive double points, which also represents the class  $c$  of the algebraic curve. Applying Theorem 5.11 we obtain

$$2p - 2 \geq c^2 + \kappa_X \cdot c = 2g(C) - 2,$$

except in the exceptional case of Theorem 5.11. In the exceptional case,  $T$  is embedded, and we can introduce a positive-negative pair of double points, and then apply the theorem.

**Theorem 5.12.** *Let  $X$  be a simply connected compact complex algebraic surface, and suppose that its canonical class  $\kappa_X$  is a basic class. Let  $C$  be an algebraic curve representing an homology class  $c \in H_2(X; \mathbf{Z})$ , and let  $T$  be a smoothly immersed 2-sphere with  $p$  positive double points also representing  $c$ . If  $p > 0$  or if  $p = 0$  and  $\kappa_X$  does not give the exceptional case of (5.11), then  $p \geq g(C)$ . Even in the exceptional case of (5.11),  $p = 0$  and  $g(C) \leq 1$ .*

This theorem for  $c^2 \geq 0$  follows from the work of Kronheimer and Mrowka. However for algebraic curves of negative self-intersection, these are the first known bounds.

#### 5.4. $SO(3)$ Donaldson invariants

We shall show how our basic recurrence relations precisely determine the relationship between the  $SU(2)$  and  $SO(3)$  Donaldson series. The following theorem has also been proved by Kronheimer and Mrowka.



**Theorem 5.13.** *Let  $X$  be a simply connected 4-manifold of simple type, and let  $c \in H_2(X; \mathbf{Z})$ . Then*

$$D_{X,c} = \exp(Q/2) \sum_{s=1}^p (-1)^{\frac{1}{2}(c^2+c \cdot \kappa_s)} a_s e^{\kappa_s}.$$

*Proof.* Write  $D_X = \exp(Q/2)K_X$ , and suppose first that  $c \in H_2(X; \mathbf{Z})$  is represented by an embedded sphere with  $c^2 = -2k$ ,  $k > 0$ . Apply Lemma 5.3 to

$$K_X = \sum_{s=1}^p a_s e^{\kappa_s}$$

to see that there are constants  $d_{i,k}$  and  $d'_{i,k}$  depending only on  $c^2$  such that on  $H_2(X)$ ,

$$\begin{aligned} & -\sinh(c) d'_{k,k} K_{X,c} \\ &= \cosh(c) \sum_{s=1}^p a_s e^{\kappa_s} \{(\kappa_s \cdot c)^{2k-1} + d_{1,k}(\kappa_s \cdot c)^{2k-3} \\ & \quad + \cdots + d_{k-1,k}(\kappa_s \cdot c)\} \\ & + \sinh(c) \sum_{s=1}^p a_s e^{\kappa_s} \{(2k-1)(\kappa_s \cdot c)^{2k-2} + d'_{1,k}(\kappa_s \cdot c)^{2k-4} \\ & \quad + \cdots + d'_{k-2,k}(\kappa_s \cdot c)^2 + d'_{k-1,k}\} \end{aligned}$$

It will be convenient to define polynomials  $A_k(z)$  and  $B_k(z)$  by

$$\begin{aligned} A_k(z) &= z^{2k-1} + d_{1,k}z^{2k-3} + \cdots + d_{k-1,k}z, \\ B_k(z) &= (2k-1)z^{2k-2} + d'_{1,k}z^{2k-4} + \cdots + d'_{k-2,k}z^2 + d'_{k-1,k}, \end{aligned}$$

so

$$\begin{aligned} (24) \quad -\sinh(c) d'_{k,k} K_{X,c} &= \cosh(c) \sum_{s=1}^p A_k(\kappa_s \cdot c) a_s e^{\kappa_s} \\ &+ \sinh(c) \sum_{s=1}^p B_k(\kappa_s \cdot c) a_s e^{\kappa_s}. \end{aligned}$$

Let  $X = E(2k)$  with  $c$  the class of a section. By Theorem 3.1, we

have

$$\begin{aligned} \mathbf{K}_{E(2k)} &= \sinh^{2k-2}(f) \\ &= \frac{1}{2^{2k-3}} \left\{ \sum_{r=1}^{k-1} (-1)^{k-1+r} \binom{2k-2}{k-1-r} \cosh(2rf) \right. \\ &\quad \left. + (-1)^{k-1} \frac{1}{2} \binom{2k-2}{k-1} \right\}, \end{aligned}$$

and

$$\begin{aligned} -\mathbf{K}_{E(2k),c} &= \cosh^{2k-2}(f) \\ &= \frac{1}{2^{2k-3}} \left\{ \sum_{r=1}^{k-1} \binom{2k-2}{k-1-r} \cosh(2rf) + \frac{1}{2} \binom{2k-2}{k-1} \right\}. \end{aligned}$$

Now substitute this into equation (24). Since  $\sinh(c) \cosh(2rf)$  and  $\cosh(c) \sinh(2rf)$  are linearly independent as functions on  $H_2(E(2k))$ , for  $r = 1, \dots, k - 1$ , we get equations

$$(25) \quad A_k(2r) = (2r)^{2k-1} + d_{1,k}(2r)^{2k-3} + \dots + d_{k-1,k}(2r) = 0,$$

$$\begin{aligned} B_k(2r) &= (2k-1)(2r)^{2k-2} + d'_{1,k}(2r)^{2k-4} + \dots + d'_{k-2,k}(2r)^2 + d'_{k-1,k} \\ &= (-1)^{k+r-1} d'_{k,k}, \\ d'_{k-1,k} &= (-1)^{k+1} d'_{k,k}. \end{aligned}$$

Note that if  $d'_{k,k} = \pm d'_{k-1,k} = 0$  then  $\pm 2, \dots, \pm(2k-2)$  together with 0 of multiplicity 2 are solutions of the polynomial equation of degree  $2k-2$ :  $B_k(z) = 0$ . Thus  $d'_{k,k} \neq 0$ .

Returning to the case of a general  $c$  represented by an embedded sphere with  $c^2 = -2k$ , let  $q(z)$  be the characteristic polynomial of  $\Delta_{c,k+1}^e$ . By Corollary

$$0 = \sum_{s=1}^p a_s \Delta_{c,k+1}^e e^{\kappa_s} = \sum_{s=1}^p a_s e^{\kappa_s} q(\kappa_s \cdot c).$$

Since the functions  $e^{\kappa_s}$  are linearly independent, the integers  $\kappa_s \cdot c$  are characteristic roots of  $\Delta_{c,k+1}^e$ , and so each  $|\kappa_s \cdot c| = 2r_s \leq 2k$ . Relabel the basic classes so that as in Theorem 5.11,  $\kappa_s \cdot c = +2k$  and  $\kappa_{s+i} = -\kappa_s$  for  $s = 1, \dots, m$ , and  $|\kappa_s \cdot c| \leq 2k-2$  for  $s = 2m+1, \dots, p$ . Then

$$\mathbf{K}_X = \sum_{s=1}^m a_s e^{\kappa_s} + \delta \sum_{s=1}^m a_s e^{-\kappa_s} + \sum_{s=2m+1}^p a_s e^{\kappa_s},$$

where  $\delta = 1$  if  $b_X^+ \equiv 3 \pmod{4}$  and  $\delta = -1$  if  $b_X^+ \equiv 1 \pmod{4}$ . Plug (25) with  $2r_s = \pm\kappa_s \cdot c$  into (24) to obtain

$$\begin{aligned}
 (26) \quad & -\sinh(c) d'_{k,k} \mathbf{K}_{X,c} = \cosh(c) A_k(2k) \left\{ \sum_{s=1}^m a_s e^{\kappa_s} - \delta \sum_{s=1}^m a_s e^{-\kappa_s} \right\} + 0 \\
 & + \sinh(c) B_k(2k) \left\{ \sum_{s=1}^m a_s e^{\kappa_s} + \delta \sum_{s=1}^m a_s e^{-\kappa_s} \right\} \\
 & - \sinh(c) \sum_{s=2m+1}^p (-1)^{k+r_s} d'_{k,k} a_s e^{\kappa_s}.
 \end{aligned}$$

Note that  $(-1)^{k+r_s} = (-1)^{\frac{c^2 + \kappa_s \cdot c}{2}}$ . If we write

$$\mathbf{K}_{X,c} = \mathbf{L} + \sum_{s=2m+1}^p (-1)^{\frac{c^2 + \kappa_s \cdot c}{2}} a_s e^{\kappa_s},$$

then

$$\begin{aligned}
 -\sinh(c) d'_{k,k} \mathbf{L} &= \cosh(c) A_k(2k) \left\{ \sum_{s=1}^m a_s e^{\kappa_s} - \delta \sum_{s=1}^m a_s e^{-\kappa_s} \right\} \\
 &+ \sinh(c) B_k(2k) \left\{ \sum_{s=1}^m a_s e^{\kappa_s} + \delta \sum_{s=1}^m a_s e^{-\kappa_s} \right\}.
 \end{aligned}$$

But by Theorem 5.11 we have

$$\sum_{s=1}^m a_s e^{\kappa_s + c} - \delta \sum_{s=1}^m a_s e^{-\kappa_s - c} = 0,$$

and therefore

$$\begin{aligned}
 (27) \quad -\sinh(c) d'_{k,k} \mathbf{L} &= \frac{1}{2} (A_k(2k) - B_k(2k)) \sum_{s=1}^m a_s e^{\kappa_s - c} \\
 &- \frac{1}{2} \delta (A_k(2k) - B_k(2k)) \sum_{s=1}^m a_s e^{-\kappa_s + c}.
 \end{aligned}$$

Next, let  $\hat{K}$  be the  $K3$  surface blown up  $2k$  times, and let  $c = e_1 + \dots + e_{2k}$ . It follows from Theorem 3.12 and Corollary 3.10 that the basic classes for  $\hat{K}$  are  $\kappa_s = \pm e_1 \pm e_2 \pm \dots \pm e_{2k}$  with

$$\mathbf{K}_{\hat{K}} = \frac{1}{p} \sum_{s=1}^p e^{\kappa_s}$$

and

$$\mathbf{K}_{\hat{K},c} = \frac{1}{p} \sum_{s=1}^p (-1)^{\frac{2k + \kappa_s \cdot c}{2}} e^{\kappa_s},$$

where  $p = 2^{2k}$ . Label the basic classes for  $\hat{K}$  so that  $\kappa_1 = -c$  and  $\kappa_2 = c$ . These are the only two basic classes for which  $|\kappa_s \cdot c| = 2k$ . In this case, (27) gives

$$\begin{aligned} -2d'_{k,k} \sinh(c) \cosh(c) &= (A_k(2k) - B_k(2k)) \sinh(-2c) \\ &= -2(A_k(2k) - B_k(2k)) \sinh(c) \cosh(c) \end{aligned}$$

i.e.,

$$d'_{k,k} = A_k(2k) - B_k(2k).$$

Thus (27) becomes

$$\begin{aligned} -\sinh(c)\mathbf{L} &= \frac{1}{2} \sum_{s=1}^m a_s e^{\kappa_s - c} - \frac{1}{2} \delta \sum_{s=1}^m a_s e^{-\kappa_s + c} \\ &= \frac{1}{2} e^{-c} \sum_{s=1}^m a_s e^{\kappa_s} - \frac{1}{2} e^c \delta \sum_{s=1}^m a_s e^{-\kappa_s} \\ &\quad + \left\{ -\frac{1}{2} e^c \sum_{s=1}^m a_s e^{\kappa_s} + \frac{1}{2} e^{-c} \delta \sum_{s=1}^m a_s e^{-\kappa_s} \right\}, \end{aligned}$$

since the term in braces is 0 by Theorem 5.11. Adding terms gives

$$-\sinh(c)\mathbf{L} = -\sinh(c) \left\{ \sum_{s=1}^m a_s e^{\kappa_s} + \delta \sum_{s=1}^m a_s e^{-\kappa_s} \right\}$$

as desired.

A similar argument works for  $c^2$  odd, and to obtain the general formula, just represent  $c$  by an immersed surface, blow up to obtain a sphere and then use the fact that the basic classes for  $X \#_k \overline{\mathbf{C}\mathbf{P}}^2$  are of the form  $K_s \pm e_1 \pm e_2 \pm \dots \pm e_k$ .

We conclude by giving a somewhat simpler characterization of the simple type condition, namely that it suffices to check the condition only on the  $SU(2)$  invariant.

**Theorem 5.14.** *Let  $X$  be a simply connected 4-manifold whose Donaldson invariant satisfies the condition  $D_X(z x^2) = 4 D_X(z)$  for all  $z \in \mathbf{A}(X)$ . Then  $X$  has simple type.*

*Proof.* We must show that the condition is also satisfied by all the  $SO(3)$  invariants  $D_{X,c}$ . Let  $c \in H_2(X; \mathbf{Z})$  and consider an immersed 2-sphere with  $p$  positive double points representing  $c$ . Blow up as usual to obtain a class  $\hat{c} = c - 2e_1 - \dots - 2e_p \in H_2(\hat{X}; \mathbf{Z})$  where  $\hat{X} = X \#_r \overline{\mathbf{C}\mathbf{P}}^2$ . Let  $\bar{X} + \hat{X} \# \overline{\mathbf{C}\mathbf{P}}^2$  with exceptional class  $e$ , and let  $\bar{c} = \hat{c} + e$ . Lemma 4.7 implies that  $D_{\bar{X}, \bar{c}}(z x^2) = 4 D_{\bar{X}, \bar{c}}(z)$  for all  $z \in \mathbf{A}(\bar{c}^\perp)$ . Embed  $H_2(X)$  in  $\langle \bar{c}^\perp \rangle$  by sending  $y$  to  $y + (y \cdot c)e$ . Using polarization identities, it suffices to consider  $z = y^n x^k \in \mathbf{A}(X)$  for arbitrary  $y \in H_2(X)$ . We shall prove

the theorem by showing that  $D_{X,c}(z x^2) = 4 D_{X,c}(z)$  by induction on  $n$ . To do this we need to use a fact proved in [14], namely that there are polynomial functions  $S_j(x)$  such that

$$D_{\hat{X} \# \overline{\mathbf{CP}}^2, \hat{c}+e}(z e^j) = D_{\hat{X}, \hat{c}}(z S_j(x)),$$

and furthermore,  $S_{2j}(x) = 0$  for all  $j$ , and  $S_1(x) = 1$ .

In case  $\deg(D_{X,c}) \equiv 0 \pmod{2}$ , begin the induction by applying Lemma 4.7 to  $X$  and  $c$  to get

$$D_{X,c}(x^{k+2}) = 4 D_{X,c}(x^k),$$

since  $x \in \mathbf{A}(c^\perp)$ . If  $\deg(D_{X,c}) \equiv 1 \pmod{2}$ , then  $\deg(D_{\hat{X}, \hat{c}}) \equiv 0 \pmod{2}$ , and we start the induction by applying Lemma 4.7 to  $\bar{X}$  and  $\bar{c}$ . Let  $y \cdot c = m$ ; so  $y + me \in \langle \bar{c}^\perp \rangle$ , and

$$\begin{aligned} D_{\bar{X}, \bar{c}}((y + me)^2 x^{k+2}) &= 2m D_{\hat{X}, \hat{c}}(y S_1(x) x^{k+2}) = 2m D_{X,c}(y x^{k+2}) \\ &= 4 D_{\bar{X}, \bar{c}}((y + me)^2 x^k) = 4 \cdot (2m) D_{X,c}(y x^k), \end{aligned}$$

since  $S_1(x) = 1$  and  $S_0(x) = S_2(x) = 0$ .

Suppose inductively that  $D_{X,c}(y^j x^{k+2}) = 4 D_{X,c}(y^j x^k)$  for all  $j < n$  and all  $k$ . Then  $D_{\bar{X}, \bar{c}}((y + me)^{n+1} x^{k+2}) = 4 D_{\bar{X}, \bar{c}}((y + me)^{n+1} x^k)$ . But

$$\begin{aligned} D_{\bar{X}, \bar{c}}((y + me)^{n+1} x^{k+2}) &= D_{\hat{X}, \hat{c}}((m y^n S_1(x) \\ &\quad + m^3 \binom{n+1}{3} y^{n-2} S_3(x) + \dots) x^{k+2}) \\ &= m D_{X,c}(y^n x^{k+2}) \\ &\quad + m^3 \binom{n+1}{3} D_{X,c}(y^{n-2} S_3(x) x^{k+2}) + \dots, \end{aligned}$$

and

$$\begin{aligned} 4 D_{\bar{X}, \bar{c}}((y + me)^{n+1} x^{k+2}) &= 4m D_{X,c}(y^n x^k) \\ &\quad + 4m^3 \binom{n+1}{3} D_{X,c}(y^{n-2} S_3(x) x^k) + \dots \end{aligned}$$

so inductively,  $D_{X,c}(y^n x^{k+2}) = 4 D_{X,c}(y^n x^k)$ .

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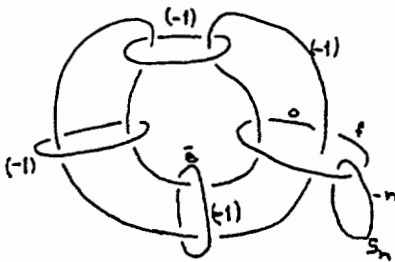


FIGURE 1.  
 $C(n)$

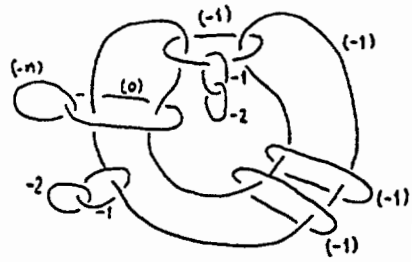


FIGURE 2.  
 $V(n)$

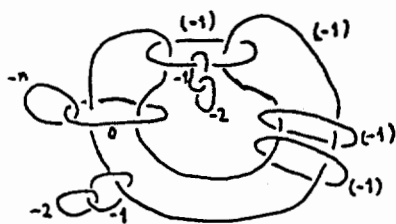


FIGURE 3.  
 $G_2 \#_f G_{n-2}$

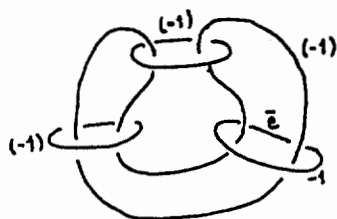


FIGURE 4.  
 $D$

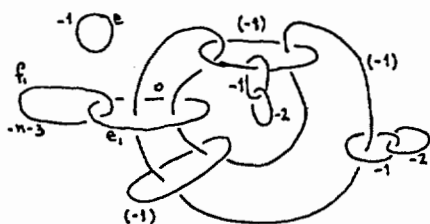


FIGURE 5.  
 $G_1 \#_f G_{n-2} \# \overline{\mathbb{C}P}^2$

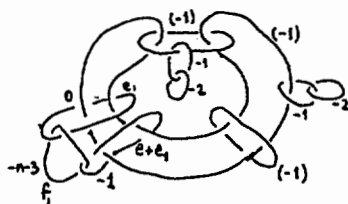


FIGURE 6.



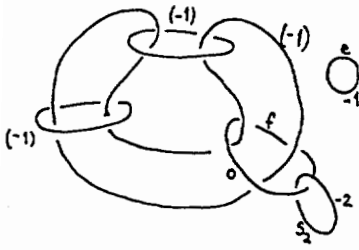


FIGURE 7.

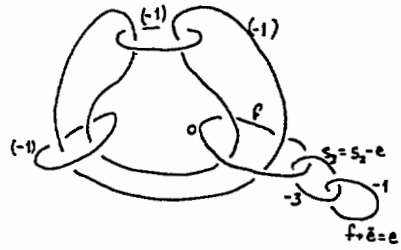


FIGURE 8.

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